MATRIX NEAR-RINGS AND GENERALIZED-DISTRIBUTIVITY

BY

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DECLARATION

This thesis is my own work. I have not submitted it, or any part of it, in a previous application for a degree.

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1 / 1 / 1989

DEDICATION

To the eternal memory of my late grandfather and first tutor — ALHAJ RASHEED AHMED ABBASI (Abba!), the priest of 'Temple of Knowledge', who guided me to the sanctuary of Mathematics.

ACKNOWLEDGEMENT

"Allah's name I begin with, extremely compassionate, the merciful"

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PREFACE

Since the construction of matrix near-rings over arbitrary near-rings in 1985, (cf. Meldrum and Van Der Walt [6]), a number of very satisfying structural results have been obtained (cf. Van Der Walt [13] and [14]) and some nice applications of the theory have been made (cf. Meyer and Van Der Walt [7] and Meyer [8]).

This encourages one to believe that matrix near-rings will play a very important role in the theory of near-rings similar to the role played by matrix rings in ring theory.

It is of interest that although there are many similarities to the ring case, one can find some striking contrasts as well. Especially noteworthy is the inequality, in general, of ideals I^+ (see definition 2.9) and I^* (see results 2.7 and 2.26) in a matrix near-ring $\mathbb{M}_n(R)$, where I is an ideal in R. This enables us to show that, unlike the ring case, not all ideals of $\mathbb{M}_n(R)$ are full (see definition 2.8 and theorem 3.45).

It may be worth pointing out that we shall not attempt a comprehensive survey of the above mentioned papers, as it has been nicely done by J. H. Meyer [8].

The major objectives of this work are:

--To extend the work on the relationship between properties of the base near-ring (and its ideals) and those of the matrix near-ring (and its ideals).

--To study d.g. (distributively-generated) and w.d. (weakly-distributive) near-rings in detail. Furthermore, to investigate how close these matrix near-rings are to matrix rings and then generalize results from matrix rings to these matrix near-rings.

An attempt is made to provide a fairly complete list of the results, on both near-rings and matrix near-rings, which we will be using in this work from time to time. Chapter 1 and chapter 2 are devoted to this purpose. Each of them has a seperate section on unpublished (so far !) results. To avoid prolonging unduly this dissertation, apart from the unpublished results, which are due to I. Roberts [10] and J. H. Meyer [8], only statements are provided. The remaining three chapters consist of my own independent research.

Although matrix near-rings have also been defined on near-rings without identity (we shall write near-rngs), it is more convenient to consider the case in which the base near-rings have an identity element. So like other researchers in matrix near-ring theory, our main interest is in near-rings with identity. But inspite of this, we deal with near-rings not necessarily with identity.

In chapter 3, we work on our first objective. We show that matrix near-rings satisfy all the additive laws which base near-rings do and when the base near-ring has an identity then it satisfies all the additive laws which the matrix near-ring does (see lemma 3.5 and theorem 3.6). Moreover, an ideal I of a near-ring with identity lies in V, a variety of additive groups, if and only if I^+ (and I^*) lies in V (see theorems 3.24 and 3.26).

It also turns out that if R has no identity, the maps $I \to I^*$ and

$I \to I^{+}$ are not necessarily injections of the set of ideals of R into the set of ideals of $\mathbb{M}_n(R)$ (see examples 3.19 and 3.20).

Chapter 4 shows the significance of d. g. near-rings. In general, the procedure for the multiplication of matrices suffers from the lack of left distributivity. In the d. g. case, one can easily escape from it, because each element of a d. g. martix near-ring can be represented as the sum of elementary matrices, $[r;\ i,\ j]$, where $r \in R$, $1 \leqslant i,\ j \leqslant n$ (see theorem 4.1). The relationship between the commutator, distributor and lower central series of R and those of $\mathbb{M}_n(R)$ is described. We also show that, under certain conditions, $I^{+} = I^{*}$ (see theorem 4.33).

Chapter 5 is devoted to the more ring like class: weakly-distributive d.g. near-rings. In a sense, we have got a close relationship between matrix rings and weakly-distributive matrix near-rings with identity. For example, we show that, like the ring case, if L is a maximal left ideal of a w.d. near-ring R with identity, and α , $\beta \in R^n - L^n$ then $(L^n : \alpha) = (L^n : \beta)$ if $\alpha \equiv \beta \pmod{L^n}$ (see theorem 5.18).

No apology is offered for the inclusion of a few results which are not precisely on matrix near-rings, as besides being of independent interest, they are useful in our work. For example, a result due to J. D. P. Meldrum (see result 1.40) is extended. We show that if R is a near-ring and G is a connected faithful R-module, then $R \in V$, a variety of additive groups, if and only if $G \in V$ (see theorem 3.17).

Notation: We shall let R denote both near-rings and rings. Matrix near-rings, the near-rings of n x n matrices over R, will be denoted by $\mathbb{M}_n(R)$. $M_n(R)$ will be the symbol for a matrix ring, the ring of n x n matrices over R.

The elements of R will be denoted by lower case Roman letters (indexed or non-indexed). Subsets, subgroups and ideals, etc. will be denoted by upper case Roman letters (indexed or non-indexed).

The elements of R^n, the direct sum of n copies of (R,+), will be denoted by lower case Greek letters (indexed or non-indexed). The vector $<0,\ldots,0>$, the zero element of R^n, will be denoted by $\underline{0}$. Subsets of R^n will sometimes be denoted by upper case Greek letters.

The elements, subsets and ideals of $\mathbb{M}_n(R)$ will be denoted by upper case script letters: $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}, \ldots$, $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$. Only zero and identity elements will be denoted by bold letters: **0** and **I**, respectively.

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CHAPTER 1

REQUISITE RESULTS ON NEAR-RINGS

For the convenience of handy reference and the sake of completeness, we present in this chapter a quick survey of requisite results on group theory and near-ring thoery.

As mentioned in the preface, only the unpublished work due to I. G. T. Roberts [10] will be proved here. References are made to all other results.

It seems appropriate to start with group theory.

Section A: Group Theory

Definition 1.1: Let $(G,+)$ be a group. If g_1 and g_2 are in G, then the commutator of g_1 and g_2 is defined as

$$(g_1,g_2) := - g_1 - g_2 + g_1 + g_2.$$

Definition 1.2: If H and K are subsets of G, then

$$(H,K) := \mathrm{Gp} \langle (h,k) : h \in H,\ k \in K \rangle$$

where $\mathrm{Gp}\langle X \rangle$ is the subgroup of G generated by X.

Result 1.3: If H and K are normal subgroups of G, then so is (H,K).

Definition 1.4: Let n be a natural number and G be a group.

$G^n := G + \ldots + G$ is the direct sum of n copies of G where

$\langle g_1,\dots,g_n\rangle + \langle h_1,\dots,h_n\rangle := \langle g_1+h_1,\dots,g_n+h_n\rangle$ for all g_i, $h_i \in G$, $i = 1, 2,\dots, n$.

<u>Result</u> <u>1.5</u>: $(G^n,G^n) = (G,G)^n$.

Proof: Let $\langle A_1,\dots,A_n\rangle \in (G,G)^n$ where $A_t \in (G,G)$ and $1 \le t \le n$. Let $g_1,\dots, g_n, h_1,\dots, h_n \in G$. We have

$\langle (g_1,h_1),\dots,(g_n,h_n)\rangle$

$= \langle -g_1 - h_1 + g_1 + h_1,\dots, -g_n - h_n + g_n + h_n\rangle$

$= -\langle g_1,\dots,g_n\rangle - \langle h_1,\dots,h_n\rangle + \langle g_1,\dots,g_n\rangle + \langle h_1,\dots,h_n\rangle$

$= (\langle g_1,\dots,g_n\rangle , \langle h_1,\dots,h_n\rangle)$.

This fact together with simple calculation implies that $\langle A_1,\dots,A_n\rangle \in (G^n,G^n)$. It follows that $(G,G)^n \subseteq (G^n,G^n)$.

To show the equality, we put $\underline{g} := \langle g_1,\dots,g_n\rangle$, $\underline{h} := \langle h_1,\dots,h_n\rangle$.

Let $\underline{X} \in (G^n,G^n)$. Then $\underline{X} = \sigma_1\underline{c}_1 +\dots+ \sigma_m\underline{c}_m$

where $\sigma_t = \pm 1$ and $\underline{c}_t$ is a commutator of $\underline{g}_t$ and $\underline{h}_t$ for $1 \leqslant t \leqslant m$.

Since $\underline{c}_t \in (G,G)^n$, the result follows immediately.

<u>Definition</u> <u>1.6</u>: The commutator subgroup of G is defined as,

$$\delta_1(G) := (G,G) := Gp\langle (g_1,g_2) : g_1, g_2 \in G\rangle.$$

Note that $g_1+g_2 = g_2+g_1+h$ where $h \in \delta_1(G)$.

<u>Definition</u> <u>1.7</u>: The higher commutator subgroups of G are defined inductively as,

$$\delta_0(G) := G \quad , \quad \delta_{m+1}(G) := (\delta_m(G), \delta_m(G)).$$

The series of higher commutator subgroups is called the derived series of G.

Result 1.8: If H is a normal subgroup of G, then G/H (quotient group of G by H) is abelian if and only if $H \supseteq \delta_1(G)$.

Definition 1.9: A group G is soluble of solubility class m if

$\delta_m(G) = \{0\}$, $\delta_{m-1}(G) \neq \{0\}$.

Definition 1.10: A chain of subgroups $\gamma_1(G)$ of G is defined inductively as

$$\gamma_1(G) := G \quad , \quad \gamma_{m+1}(G) := (\gamma_m(G), G).$$

The descending central series of G is the normal series

$$G = \gamma_1(G) \supseteq \gamma_2(G) \supseteq \ldots\ldots$$

Definition 1.11: A group G is nilpotent of nilpotency class m if

$$\gamma_{m+1}(G) = \{0\} \quad , \quad \gamma_m(G) \neq \{0\}.$$

Remark: All abelian groups are nilpotent and all nilpotent groups are soluble. But one can find nilpotent groups which are not abelian and soluble groups which are not nilpotent.

Result 1.12: (page 56 Robinson [11])

A variety is an equationally defined class of groups. More precisely, if W is a set of words in x_1, $x_2, \ldots$, the class of all groups G such that W(G) = 0, is called the variety V.

(2.3.4 Robinson [11])

Every variety is closed with respect to forming subgroups, images and subcartesian proud~~u~~cts of its elements.

Section B: Near-Rings

We follow Meldrum [5] in our notations and definitions, although he uses left near-rings. Both Meldrum [5] and Pilz [9] are used as our source of standard results.

Definition 1.13 (1.1 [5]) A non-empty set R with two binary operations $+$ and $.$ is a right near-ring $(R,+,.)$ if

(1): $(R,+)$ is a group (not necessarily abelian)

(2): $(R,.)$ is a semigroup

(3): the operation $.$ is right distributive over the operation $+$, that is,

$(y + z).x = y.x + z.x,$ for all x, y, z in R.

$(R,+,.)$ is called a left near-ring if it satisfies (1), (2) and, naturally enough, the operation multiplication is left distributive over the operation addition.

It is customary to omit the symbol $.$ for multiplication and abbreviate $(R,+,.)$ by R. 0 is the additive identity of R and 1 (if it exists) is the multiplicative identity of R.

Example 1.14 (1.2 [5])

The most common example of a near-ring is $(M(G),+,.)$ where

$$M(G) := \{\theta: G \to G\},$$

addition is defined pointwise and multiplication is composition of maps.

Definition 1.15 (1.7 [5])

A near-ring R is abelian if $(R,+)$ is abelian.

Definition 1.16 (1.11 [5])

If x is in R and $x0 = 0$ then R is called a zero-symmetric near-ring. Note that $0x = 0$ for all x in R (see 1.10 [5]).

Definition 1.17 (1.28 [5])

Let R be a near-ring and H be a subset of R. H is called a right R-subgroup of R if

(1): $(H,+)$ is a subgroup of $(R,+)$

(2): $HR \subseteq H$.

H is called a left R-subgroup of R if

(1): $(H,+)$ is a subgroup of $(R,+)$

(2): $RH \subseteq H$.

H is called a two-sided R-subgroup of R if it is both a left and right R-subgroup.

Definition 1.18 (1.21 [5])

Let R be a near-ring and I be a subset of R. I is called a right ideal of R if

(1): $(I,+)$ is a normal subgroup of $(R,+)$

(2): $IR \subseteq I$.

I is called a left ideal of R if

(1): $(I,+)$ is a normal subgroup of $(R,+)$

(2): $x(a + y) - xy \in I$ for all $x, y \in R$ and $a \in I$.

I is called an ideal of R if it is both a left and a right ideal.

Definition 1.19 (1.16 [5])

Let R and T be two near-rings. A mapping θ from R to T is called a near-ring homomorphism if

(1): $\theta(x + y) = \theta x + \theta y$

(2): $\theta(xy) = (\theta x)(\theta y)$

for all x, y in R.

Definition 1.20 (2.1 [5])

Let R be a near-ring and $(G,+)$ be a group. If there is a near-ring homomorphism θ from R to $M(G)$, then G is called a (left) R-module. θ is called a representation of R, which is f. thful if $\ker \theta = \{0\}$.

Example 1.21 (2.9 [5])

We let $_RR$ denote the additive group $(R,+)$. An R-module structure for $_RR$ can now be given. We define

$$\lambda : R \to M(_RR)$$

by

$$\lambda(r)x := rx$$

where $r \in R$, $x \in {}_RR$ and rx is the product in R.

It can be checked easily that λ is a near-ring homomorphism. It is called the left regular representation.

Definition 1.22 (2.10 [5])

Let R be a near-ring and G be an R—module. A subgroup H of G is called an R-submodule of G if $RH \subseteq H$.

Definition 1.23 (2.10 [5])

A normal subgroup $(H,+)$ of G is called an R-ideal of G if $x(g + h) - xg \in H$ for all $g \in G$, $h \in H$ and $x \in R$.

Result 1.24 (2.12 [5])

Let R be a near-ring and G an R-module. Then

$0_R\, g = 0_G$, for all $g \in G$.

Definition 1.25 (2.30 [5])

Let G be an R-module. If H_1 and H_2 are subsets of G, then

$$(H_1:H_2) := \{x \in R: xH_2 \subseteq H_1\}.$$

Note that if $H_2 = \{k\}$, then we write $(H_1: k)$ in place of $(H_1:\{k\})$. Similarly if $H_1 = \{h\}$, then $(\{h\}:H_2)$ is denoted by $(h:H_2)$. $(0:H_2)$ is called the annihilator of H_2 in R, and is denoted by $\mathrm{Ann}_R H_2$.

Result 1.26 (2.31 [5])

Let G be an R-module, H_1 and H_2 be subsets of G. If H_1 is an R-ideal, then $(H_1:H_2)$ is a left ideal of R.

Definition 1.27 (1.36a [9])

An R-module G is simple if and only if it has no non-trivial R-ideals.

Definition 1.28 (1.35 and 1.36b [9])

Let R be a zero-symmetric near-ring. An R-module G is R-simple if and only if G has no R-submodules except G and $\{0_G\}$.

Definition 1.29 (1.39 [9])

A maximal ideal of R is an ideal which is maximal in the set of all proper ideals

Note (page 86 [9]): Let R be a zero-symmetric near-ring and let L be a maximal left ideal of R. L is called a strictly maximal left ideal of R if it is also maximal as a left R-subgroup.

<u>Result</u> <u>1.30</u> (1.40 [9])

I is a maximal ideal of R if and only if R/I is simple.

H is a maximal R-ideal in G if and only if G/H is simple.

<u>Result</u> <u>1.31</u> (2.16 [5])

Let R be a zero-symmetric near-ring and G be an R-module. Then every R-ideal of G is an R-submodule.

<u>Definition</u> <u>1.32</u> (3.1 [5])

Let R be a near-ring and G be an R-module. If there exists g in G such that Rg = G, that is,

$$G = \{xg : x \in R\}$$

then G is called a monogenic R-module.

The next definition due to Van Der Walt [13] gives a generalization of monogenic R-modules.

<u>Definition</u> <u>1.33</u> (3.1 [13])

An R-module G is called a connected R-module if for any g_1, g_2 in G, there are g in G and x, y in R such that $g_1 = xg$ and $g_2 = yg$.

<u>Result</u> <u>1.34</u> (3.2 [13])

Let G be a connected R-module. If $g_1, \dots, g_k$ are any $k \geq 2$ elements of G, then there are $r_1, \dots, r_k$ in R and g in G such that $g_t = r_t g$ where $t = 1, \dots, k$.

<u>Definition</u> <u>1.35</u> (3.1b [9])

An R-module G is strongly monogenic if G is monogenic and for all g in G, either $Rg = \{0\}$ or $Rg = G$.

<u>Definition</u> <u>1.36</u> (3.4 [5])

A monogenic R-module G is an R-module of type 0 if G is simple, that is, it has no non-trivial proper R-ideals; G is an R-module of type 1 if G is simple and strongly monogenic; G is an R-module of type 2 if G is non-trivial and has no non-trivial proper R-submodule.

<u>Result</u> <u>1.37</u> (3.5 [5])

If R is a near-ring with identity and all R-modules considered are unitary, then type 1 modules are of type 2 and so the two types are equivalent .

<u>Definition</u> <u>1.38</u> (3.6 [5])

Let R be a near-ring and G be an R-module. For $\nu = 0, 1, 2$, R is called ν-primitive on G if G is faithful and of type ν.

<u>Result</u> <u>1.39</u> (12.9 [5])

Let R be a near-ring and G be an R-module. If $w(x_1, \ldots, x_p)$ is a word in p variables $x_1, \ldots, x_p$, then

$$w(r_1, \ldots, r_p)g = w(r_1 g, \ldots, r_p g)$$

where $r_1, \ldots, r_p \in R$ and $g \in G$.

<u>Result</u> <u>1.40</u> (12.10 [5])

Let G be a faithful R-module. If G lies in V, a variety of additive groups, then so does R.

<u>Definition</u> <u>1.41</u> (5.1 [5])

A map which assigns to each near-ring R an ideal $R(R)$ is a radical map if for every near-ring R and every homomorphism $\theta : R \to T$, we have

(1): $\mathcal{R}(R/\mathcal{R}(R)) = \{0\}$.

(2): $\theta(\mathcal{R}(R)) \subseteq \mathcal{R}(\theta(R))$.

Definition 1.42 (5.4 [5])

Let R be a near-ring. For $\nu = 0, 1, 2$

$$J_\nu(R) := \cap \{\mathrm{Ann}_R G : G \text{ is an R-module of type } \nu\}$$

is called the ν-radical of R.

Result 1.43 (5.5 [5])

For $\nu = 0, 1, 2$, the map J_ν is a radical map.

Result 1.44 (5.8 [5])

Let R be a near-ring with identity. Then $J_1(R) = J_2(R)$.

Definition 1.45 (5.9 [5])

Let R be a near-ring and L be a left ideal of R. L is called a ν-maximal left ideal if R/L is an R-module of type ν, $\nu = 0, 1, 2$ (via the left regular representation: $x(y + L) := xy + L$).

Definition 1.46 (3.20 [9])

A left ideal L of R is called modular if and only if there exists e in R such that $x - xe \in L$ for all x in R. e is called a right identity modulo L (since for all x in R, $xe \equiv x \pmod{L}$.

Definition 1.47 (3.28 [9])

A left ideal L of R is called a ν-modular left ideal if it is a modular and ν-maximal left ideal.

Definition 1.48 (5.25 [5])

Let R be a near-ring. We define $J_{1/2}(R)$ by

$$J_{1/2}(R) := \cap \{K : K \text{ is a 0-modular left ideal of } R\}.$$

In other words,

$$J_{1/2}(R) := \cap \{K: K \text{ is a maximal modular left ideal of } R\}.$$

<u>Result</u> <u>1.49</u> (5.26 [5])

Let R be a near-ring. Then $J_{1/2}(R)$ is a left ideal of R and

$$J_0(R) \subseteq J_{1/2}(R) \subseteq J_1(R) \subseteq J_2(R).$$

<u>Result</u> <u>1.50</u> (5.35 [5])

Let R be a near-ring. Then $J_{1/2}(R)$ contains all nil ideals of R.

<u>Definition</u> <u>1.51</u> (6.15 [5])

Let R be a near-ring and I, J, P be ~~the~~ ideals of R. P is called a prime ideal if whenever $IJ \subseteq P$ then $I \subseteq P$ or $J \subseteq P$.

R is called a prime near-ring if {0} is a prime ideal.

<u>Result</u> <u>1.52</u> (6.31 [5])

Any ν-primitive ideal is a prime ideal, for $\nu = 0, 1, 2$.

Section C: Distributively-Generated Near-Rings

Although a right near-ring R suffers from the lack of a left distributive law, there may be some elements in R which satisfy both distributive laws.

Definition 1.53 (9.1 [5])

An element d in R is called a distributive element of R if

$$d(x + y) = dx + dy$$

for all x and y in R.

Definition 1.54 (9.27 [5])

A right near-ring R is called distributive if

$$r(x + y) = rx + ry$$

for all r, x and y in R.

Result 1.55 (9.5 [5])

The set of all distributive elements of a near-ring R, denoted by R_d, forms a semigroup under multiplication.

Definition 1.56 (9.8 [5])

Let $(S,.) \subseteq (R_d,.)$. A near-ring R is called a distributively-generated near-ring, abbreviated as, d.g.near-ring, if (R,+) is generated as a group by (S,.).

Because of the importance of S, a d.g. near-ring R is, generally, denoted by (R,S).

0 is assumed to be in S, since (S,.) is a semigroup of distributive elements which generates (R,+) if and only if $S \cup \{0\}$ does the same.

Result 1.57 (9.12 [5])

Let S be a semigroup of distributive elements of a not necessarily zero-symmetric near-ring R. Then any d.g. near-ring (R,S) is zero-symmetric.

Result 1.58 (9.26 [5])

Let R be an abelian d.g. near-ring. Then (R,S) is a ring.

Result 1.59 (9.29 [5])

A d.g. near-ring (R,S) is distributive if and only if the elements of $(R^2,+)$ commute with each other.

Result 1.60 (9.30 [5])

A distributive near-ring with identity is a ring.

Result 1.61 (4.1.1 Frohlich [2])

If I_1 and I_2 are ideals of (R,S), then (I_1,I_2) is an ideal of (R,S).

Note: For a non d.g.near-ring R, (I_1,I_2) need not be an ideal of R (see example 4.8 [10]).

Result 1.62 (9.34 [5])

Let (R,S) be a d.g. near-ring. Then for all $m \geq 0$, $\delta_m(R)$ and $\gamma_{m+1}(R)$ are ideals of R.

Definition 1.63 (9.35 [5])

Let x, y, z be in (R,S). We define the distributor of x over y and z as

$$(x;y,z) := x(y + z) - xz - xy.$$

If X, Y, Z are subsets of (R,S) then

$$(X;Y,Z) := \mathrm{Gp}\langle (x;y,z) : x \in X,\ y \in Y,\ z \in Z\rangle.$$

Definition 1.64 (9.35 [5])

Let I be an ideal of (R,S). We define the distributor series by

$$D^0(I) := I,$$

$$D^1(I) := \mathrm{Gp}\ \langle (R;I,I)\rangle^R,$$

$$D^{m+1}(I) := \mathrm{Gp}\ \langle (R;D^m(I),D^m(I))\rangle^R,$$

where $\mathrm{Gp}\ \langle X\rangle^R$ is the normal subgroup of R generated by X.

Result 1.65 (9.38 [5])

Let X, Y, Z be subsets of (R,S).

(1) if X is a left R-subgroup then so is (X;Y,Z) and hence so is (R;Y,Z).

(2) if Y and Z are right R-subgroups then so is (X;Y,Z).

(3) $\mathrm{Gp}\ \langle (R;Y,Z)\rangle^R$ is a left ideal.

(4) if Y, Z are right R-subgroups then $\mathrm{Gp}\ \langle (R;Y,Z)\rangle^R$ is an ideal.

Definition 1.66 (9.40 [5])

A d.g. near-ring (R,S) is weakly-distributive (w.d. in short) if $D^m(R) = \{0\}$ for some integer m.

Definition 1.67 (9.40 [5])

We define a weakly-distributive series for R as a series of ideals

$$R = I_0 \geq I_1 \geq \ldots \geq I_n = \{0\}$$

such that all the elements of R/I_j are distributive with respect to all pairs of elements in I_{j-1}/I_j .

Result 1.68 (9.42 [5])

Let Y, Z be subsets of (R,S). Then

$$\text{Gp} <(R;Y,Z)>^R = \text{Gp} <(RY,RZ)>^R .$$

If Y, Z are left R-subgroups and R has an identity, then

$$\text{Gp} <(R;Y,Z)>^R = \text{Gp} <(Y,Z)>^R .$$

Result 1.69 (9.45 [5])

Let (R,S) be a d.g. near-ring with $R^2 = R$. Then $D^m(R) = \delta_m(R)$, for all $m \geq 0$.

Result 1.70 (9.46 [5])

Let (R,S) be a d.g. near-ring with $R^2 = R$. Then (R,S) is w.d. if and only if (R,+) is soluble.

Result 1.71 (9.49 [5])

If (R,S) is a d.g. near-ring with (R,+) soluble, then $\delta_1(R)$ is multiplicatively nilpotent.

Result 1.72 (9.55 [5])

Let (R,S) be a w.d. near-ring. Then $D^1(R)$ is multiplicatively nilpotent.

Result 1.73 (13.10 [5])

If (R,S) is d.g. and $X \subseteq R$, then the ideal I of (R,S) generated by X is the normal subgroup of (R,+) generated by

$$SXR := \{sxr,\ sx,\ xr,\ x;\ x \in X,\ s \in S,\ r \in R\} .$$

Result 1.74 (14.22 [5])

If (R,S) is a d.g. near-ring with (R,+) soluble, then all maximal left ideals of (R,S) are strictly maximal.

Result 1.75 (14.23 [5])

If (R,S) is a d.g.near-ring with (R,+) soluble, then

$$J_0(R) = J_2(R) \supseteq \delta_1(R).$$

Section D: c-Nilpotent and d.w.d. Near-Rings

Another type of generalized distributivity in d.g. near-rings, d-weak distributivity, was defined in Roberts [10].

A d-weakly distributive near-ring (d.w.d.near-ring, in short) comes between distributive and weakly-distributive near-rings. We first define $\bar{d}$-distributor and $\underline{d}$-distributor series of ideals of (R,S).

Definition 1.76 (7.1 [10])

Let I be an ideal of (R,S). The $\bar{d}$-distributor series of ideals in (R,S) is defined as follows:

$d^0(I) := I$,

$d^1(I) := \mathrm{Gp}\langle (r;a,b) : a,b \in I,\ r \in R\rangle^R$,

$d^{m+1}(I) := \mathrm{Gp}\langle (R;I,d^m(I))\rangle^R$,

where $\mathrm{Gp}\langle X\rangle^R$ is the normal subgroup of R generated by X.

Result 1.77 (7.6 [10])

$d^m(I)$ is an ideal of (R,S).

Proof: We use induction on m. The case m = 0 is obvious. Let m = 1. Since Gp $<(R;I,I)>^R$ is an ideal, by result 1.65, so $d^1(I)$ is an ideal. Suppose $d^m(I)$ is an ideal then again by result 1.65, Gp $<(R;I,d^m(I))>^R$ is an ideal, that is, $d^{m+1}(I)$ is an ideal.

Definition 1.78 (9.6 [10])

Let I be an ideal of a d.g. near-ring (R,S). Define $d_0(I) := \{0\}$, and $d_1(I)$ as the maximal ideal of (R,S) contained in I such that $(R;I,d_1(I)) \subseteq d_0(I) = \{0\}$.

If $d_m(I)$ is an ideal of (R,S) such that $d_m(I)$ contained in I, then we define $d_{m+1}(I)$ to be the maximal ideal of (R,S) contained in I such that $(R;I,d_{m+1}(I)) \subseteq d_m(I)$.

Definition 1.79 (7.5 [10])

A d.g. near-ring (R,S) is $\bar{d}$.w.d. if $d^m(R) = \{0\}$ for some positive integer m.

Definition 1.80 (9.8 [10])

A d.g. near-ring (R,S) is $\underline{d}$.w.d. if $d_m(R) = R$ for some positive integer m.

Result 1.81 (9.9 [10])

Let (R,S) be a $\bar{d}$.w.d. near-ring then it is $\underline{d}$.w.d.

Proof: We first show that $d^{m-i}(R) \subseteq d_i(R)$ where $0 \leq i \leq m$, $m \geq 0$. We use induction on i. The case i = 0 is trivial, as $d^m(R) = d_0(R) = \{0\}$, by $\bar{d}$-weak distributivity of (R,S) and by definition 1.78.

Suppose that $d^{m-i}(R) \subseteq d_i(R)$, then by definition,

$$d^{m-i}(R) = \mathrm{Gp} <(R;R,d^{m-i-1}(R))>^R$$

$$\subseteq d_i(R), \text{ by the induction hypothesis.}$$

Now since $d_{i+1}(R)$ is the maximal ideal of (R,S) such that

$\mathrm{Gp} <(R;R,d_{i+1}(R))> \subseteq d_i(R)$ therefore $d^{m-i-1}(R) \subseteq d_{i+1}(R)$.

The result now follows by induction.

Putting $m = i$ in $d^{m-i}(R) \subseteq d_i(R)$, we get $d^0(R) \subseteq d_m(R)$. That is,

$R \subseteq d_m(R)$ as $d^0(R) = R$, by definition 1.76. Hence $R = d_m(R)$ and

(R,S) is $\underline{d}$.w.d., by definition 1.80.

<u>Result</u> <u>1.82</u> (9.9 [10])

(R,S) is $\bar{d}$.w.d. if and only if (R,S) is $\underline{d}$.w.d.

Proof: The necessity of the conditions follows from the above result. To prove the sufficiency, we need to show that if (R,S) is $\underline{d}$.w.d. then $d^{m-i}(R) \subseteq d_i(R)$ where $0 \leq i \leq m$.

We use reverse induction on i. If $i = m$ then $d^0(R) = d_m(R) = R$, by definition 1.76 and $\underline{d}$ weak-distributivity of (R,S).

Now assume that $d^{m-i}(R) \subseteq d_i(R)$.

Since $d^{m-i+1}(R) = \mathrm{Gp} <(R;R,d^{m-i}(R))>^R$, by definition 1.76,

$\subseteq \mathrm{Gp} <(R;R,d_i(R))>^R$, by the induction hypothesis,

$\subseteq d_{i-1}(R)$, by definition 1.78,

the result now follows for all i, where $0 \leq i \leq m$.

Put $i = 0$ in $d^{m-i}(R) \subseteq d_i(R)$. We get $d^m(R) \subseteq d_0(R) = \{0\}$, by definition 1.78. That is, $d^m(R) = \{0\}$ and hence (R,S) is $\bar{d}$.w.d.

Definition 1.83 (4.17 [10])

Let R be a near-ring and I be a subgroup of $(R,+)$. The descending central series is defined as follows:

$Z^0(I) := I, \quad Z^1(I) := (I,I), \quad Z^{m+1}(I) := (I,Z^m(I)).$

Result 1.84 (4.19 [10])

Let (R,S) be a d.g. near-ring. If I is an ideal in (R,S), then $Z^m(I)$ is also an ideal in (R,S).

Proof: The case $m = 0$ is trivial. If $m = 1$, the result follows immediately from result 1.61. Suppose that $Z^m(R)$ is an ideal of (R,S) then $Z^{m+1}(R) = (R,Z^m(R))$ is an ideal of (R,S), by result 1.61. The result now follows by induction.

Having faced problems in defining the ascending central series for an ideal I of a near-ring R in such a way that the terms of the series will be ideals, Fröhlich [2] defined the series $Z_m(R)$ as follows:

Definition 1.85 (9.1 [10])

Let I be an ideal of a near-ring R, and Z(I) be the maximal ideal of R contained in the centre of I. Note that the centre of I need not be an ideal of R. We define the ascending central ideal series of I as follows:

$Z_0(I) := \{0\}, \quad Z_1(I) := Z(I).$

Assume that $Z_m(I)$ has already been defined as an ideal of R contained in I. Now consider the quotient near-ring $\hat{R} = R/Z_m(I)$, then in $\hat{R}$ the ideal I has image $\hat{I} = I/Z_m(I)$.

We define $Z_{m+1}(I)$ to be the maximal ideal of R contained in I whose image under the natural mapping $R \to \hat{R}$ is contained in the centre of $\hat{I}$.

<u>Definition</u> <u>1.86</u> (4.17 [10])

If there is a positive integer m such that $Z^m(R) = \{0\}$ then R is said to be $\bar{c}$-nilpotent.

<u>Definition</u> <u>1.87</u> (9.3 [10])

A near-ring R is said to be $\underline{c}$ — nilpotent if there is a positive integer m such that $Z_m(R) = R$.

In fact, in the d.g. case, $\bar{c}$-nilpotence and $\underline{c}$-nilpotence are equivalent.

<u>Result</u> <u>1.88</u> (9.4 [10])

If (R,S) has a finite series

$$R = I_0 \geq I_1 \geq \dots \geq I_m = \{0\}$$

of ideals I_t of (R,S) such that $(R,I_t) \leq I_{t+1}$ where $0 \leq t \leq m-1$, then

(1) $Z^i(R) \leq I_i$ where $0 \leq i \leq m$.

(2) $I_{m-i} \leq Z_i(R)$ where $0 \leq i \leq m$.

Proof: (1) The result follows easily from definition 1.83 and induction on i.

(2) We use induction on i. The case $i = 0$ is obvious as $I_m = Z_0(R) = \{0\}$, by hypothesis and definition 1.72.

Let $I_{m-i} \leq Z_i(R)$. If $x \in I_{m-i-1}$ and $y \in R$, then

$$\begin{aligned}(x,y) &\in (I_{m-i-1}, R)\\ &= (R, I_{m-i-1})\\ &\leq I_{m-i}, \text{ by hypothesis,}\\ &\leq Z_i(R), \text{ by the induction hypothesis.}\end{aligned}$$

So that x commutes with each element of R mod $(Z_i(R))$. Thus $I_{m-i-1}/Z_i(R) \leq Z\,(R/Z_i(R))$, where $Z(R)$ is the centre of R.

Since I_{m-i-1} is an ideal, and $Z_{i+1}(R)$ is the maximal ideal of R such that $Z_{i+1}(R)/Z_i(R) \leq Z\,(R/Z_i(R))$, by definition 1.85, therefore $I_{m-i-1} \leq Z_{i+1}(R)$, as required.

<u>Result</u> <u>1.89</u> (9.5 [10])

A d.g. near-ring (R,S) is $\bar{c}$-nilpotent if and only if it is $\underline{c}$-nilpotent .

Proof: Let (R,S) be $\bar{c}$-nilpotent. Put $m = i$ in the result 1.88 (2). We get $I_0 \leq Z_m(R)$, that is, $R \leq Z_m(R)$ as (R,S) has a central series of ideals I_i of (R,S) with $I_0 = R$. Therefore $R = Z_m(R)$ and hence R is $\underline{c}$-nilpotent. Conversely, suppose (R,S) is $\underline{c}$-nilpotent.

Putting $i = m$ in result 1.88 (1), we get

$Z^m(R) \leq I_m = \{0\}$. This forces (R,S) to be $\bar{c}$-nilpotent.

<u>Result</u> <u>1.90</u> (7.15 [10])

Let (R,S) be a d.g. near-ring. Then $d^m(R) \leq Z^m(R)$ for all $m \geq 0$.

Proof: We use induction on m. The case m = 0 is obvious, as

$d^0(R) = Z^0(R) = R$, by definitions 1.76 and 1.83.

Let m = 1. Then

$d^1(R) = \mathrm{Gp} <(R;R,R)>^R$, by definition 1.76,

$= \mathrm{Gp} <(R\cdot R,R\cdot R)>^R$, by result 1.68,

$\leq \mathrm{Gp} <(R,R)>^R$, as $R^2 \leq R$,

$= (R,R)$, as (R,R) is a normal subgroup of R, by result 1.3,

$= Z^1(R)$, by definition 1.83.

Next assume that $d^m(R) \leq Z^m(R)$, then

$d^{m+1}(R) = \mathrm{Gp} <(R;R,d^m(R))>^R$

$= \mathrm{Gp} <(R\cdot R,R\cdot d^m(R))>^R$, by result 1.68,

$\leq \mathrm{Gp} <(R,d^m(R))>^R$, as $d_m(R)$ is an ideal of (R,S), by result 1.77,

$\leq \mathrm{Gp} <(R,Z^m(R))>^R$, by the induction hypothesis,

$= (R,Z^m(R))$, as $(R,Z^m(R)))$ is a normal subgroup of R, by result 1.3

$= Z^{m+1}(R)$, by definition 1.83.

The result now follows by induction.

<u>Result</u> <u>1.91</u> (7.16 [10])

If (R,S) is $\bar{c}$-nilpotent then it is $\bar{d}$.w.d.

Proof: Let (R,S) be $\bar{c}$-nilpotent. Since

$d^m(R) \leq Z^m(R)$, by result 1.90,

$= \{0\}$, by $\bar{c}$-nilpotency,

therefore $d^m(R) = \{0\}$. Hence (R,S) is $\bar{d}$.w.d.

<u>Result</u> <u>1.92</u> (7.18 [10])

Let (R,S) be a d.g. near-ring with a left identity. Then

$Z^m(R) = d^m(R)$ for all $m \geq 0$.

Proof: We use induction on m. The case m = 0 is obvious, as

$Z^0(R) = d^0(R) = R$. If m = 1, then

$Z^1(R) = (R,R)$

$= Gp <(R,R)>^R$

$= Gp <(R;R,R)>^R$, by result 1.68,

$= d^1(R)$.

We next assume that $Z^m(R) = d^m(R)$. We aim to show that

$Z^{m+1}(R) = d^{m+1}(R)$.

$Z^{m+1}(R) = (R,Z^m(R))$, by definition 1.83,

$= Gp <(R,Z^m(R))>^R$

$= Gp <(R;R,Z^m(R))>^R$, by result 1.68,

$= Gp <(R;R,d^m(R))>^R$, by the induction hypothesis,

$= d^{m+1}(R)$.

The result now follows by induction.

<u>Result</u> <u>1.93</u> (7.19 [10])

Let (R,S) be a d.g. near-ring with a left identity. Then (R,S) is $\bar{c}$-nilpotent if and only if it is $\bar{d}$.w.d.

Proof: The result follows immediately from result 1.92, definitions 1.79 and 1.86.

<u>Result</u> <u>1.94</u> (9.14 [10])

Let (R,S) be a d.g. near-ring with a left identity. Then (R,S) is $\underline{c}$-nilpotent if and only if it is $\underline{d}$.w.d.

Proof: (R,S) is $\underline{c}$-nilpotent

if and only if (R,S) is $\bar{c}$-nilpotent, by result 1.89,

if and only if (R,S) is $\bar{d}$.w.d., by result 1.93,

if and only if (R,S) is $\underline{d}$.w.d., by result 1.82.

CHAPTER 2

REQUISITE RESULTS ON MATRIX NEAR-RINGS

Our aim in this chapter will be to collect all the requisite results on matrix near-rings.

There are many instances in which properties of a base near-ring R are carried over to a matrix near-ring $\mathbb{M}_n(R)$ and vice-versa. But it turns out that the process of carrying over properties of $\mathbb{M}_n(R)$ to R is, sometimes, obstructed if R does not have an identity. For example, $\mathbb{M}_n(R)$ may be a rng (ring with out identity) even though R is not a rng. (see the remark preceding theorem 4.12). This shows that the existence of an identity element in R is an important condition. Hence, for the sake of clarity, we state all the results on matrix near-rings with identity in a seperate section.

As mentioned in the preface, only the unpublished work due to J. H. Meyer [8] will be proved here. References are made to all other results.

Section A: Some Results on Matrix Near-Rings

Let R be a right near-ring and $n \in \mathbb{N}$, the set of all natural numbers . The direct sum of n copies of the group (R, +) is denoted by R^n. Note that (R, +) is not necessarily abelian. The elements of

R^n are thought of as column vectors, but for typographical reasons, we write them in transposed form with pointed brackets. For example, $<x_1, x_2, \ldots, x_n> \in R^n$.

The n x n matrices will be defined as functions from R^n to R^n. First we recall the following familiar embeddings which will be used to define these matrices.

Let R be a near-ring with identity. R can be embedded into the near-ring M(R) of all mappings of (R, +) into itself, by means of the rules $x \to f^x$, where $f^x(y) = xy$ for all y in R.

Let R be a near-ring. The symbol ∞ can be joined to the group (R, +) and we can obtain the group with infinity $(R_\infty, +)$. Then $M(R_\infty)$, the set of all functions from R_∞ to R, is a near-rng under pointwise addition and composition. We can now embed R into $M(R_\infty)$, by means of the rule $x \to f^x$ where

$$f^x(y) = \begin{matrix} xy & \text{if } y \in R \\ x & \text{if } y = \infty \end{matrix} .$$

We are now able to introduce the functions

$$f^x_{ij} : R^n \to R^n$$

where

$$f^x_{ij} = \iota_i \; f^x \; \pi_j$$

for $1 \leqslant i, j \leqslant n$, $x \in R$ and the symbols ι_j and π_j denote the j-th co-ordinate injection and projection, respectively.

In the ring case, f^x_{ij} corresponds to a matrix with x in the (i,j)-th position and 0 elsewhere.

Definition 2.1 (2.1 [6])

The near-ring of n x n matrices over R, denoted by $\mathbb{M}_n(R)$, is the subnear-ring of $M(R^n)$ generated by the set $\{f^x_{ij}: x \in R,\ 1 \leqslant i,j \leqslant n\}$. The elements of $\mathbb{M}_n(R)$ will be referred to as n x n matrices over R.

Remark: We use this definition even when R does not have an identity. So R is not necessarily embedded in $\mathbb{M}_n(R)$.

In Meldrum and Van Der Walt [6], $\mathbb{M}_n(R)$ is defined as a subnear-ring of $M(R^n_\infty)$.

Definition 2.2 (2 [14])

The set $\mathbb{E}_n(R)$ of matrix expressions is the subset of the free semigroup over the alphabet of symbols

$$\{f^x_{ij} : x \in R,\ 1 \leqslant i,j \leqslant n\} \cup \{(\ ,\ ),\ +\}$$

recursively defined by the following rules:

(1) $f^x_{ij} \in \mathbb{E}_n(R)$ for $1 \leqslant i,\ j \leqslant n$ and all $x \in R$.

(2) If $\mathcal{X}, \mathcal{Y} \in \mathbb{E}_n(R)$, then $\mathcal{X} + \mathcal{Y} \in \mathbb{E}_n(R)$.

(3) If $\mathcal{X}, \mathcal{Y} \in \mathbb{E}_n(R)$, then $(\mathcal{X})(\mathcal{Y}) \in \mathbb{E}_n(R)$.

Any matrix $\mathcal{X}$ can be represented as an expression involving only the f^x_{ij}. The length, $l(\mathcal{X})$, of such an expression is simply the number of f^x_{ij} in it. The weight, $w(\mathcal{X})$, of $\mathcal{X}$ is the length of an expression of minimal length for $\mathcal{X}$.

It is clear that if $\mathfrak{X}$ is represented by an expression of length $w(\mathfrak{X}) \geqslant 2$, then there are two possibilities:

$\mathfrak{X} = \mathfrak{X}_1 + \mathfrak{X}_2$ or $\mathfrak{X} = \mathfrak{X}_1\mathfrak{X}_2$, where $w(\mathfrak{X}_1)$, $w(\mathfrak{X}_2) < w(\mathfrak{X})$.

Note that, for typographical reasons, we use, henceforth, the symbol $[x;i,j]$ for f^x_{ij}.

<u>Result</u> <u>2.3</u> (3.1 [6])

For all i, j, k, $l \in \{1, 2, \ldots, n\}$ and x, y, z, $x_1, \ldots, x_n \in R$, we have

(1) $[x;i,j] + [y;i,j] = [x + y;i,j]$,

(2) $[x;i,j] + [y;k,l] = [y;k,l] + [x;i,j]$ if $i \neq k$,

(3) $[x;i,j] \,.\, [y;k,l] = \begin{matrix} [xy;i,l] & \text{if } j = k \\ [x0;i,l] & \text{if } j \neq k \end{matrix}$

(4) $-[x;i,j] = [-x;i,j]$,

(5) $[x;i,j]\,([x_1;1,l_1] + \ldots + [x_n;n,l_n]) = [x;i,j]\,[x_j;j,l_j]$
$= [xx_j;i,l_j]$

where $l_m \in \{1, 2, \ldots, n\}$, $m = 1, 2, \ldots, n$,

(6) x is zero-symmetric in R if and only if $[x;i,j]$ is zero-symmetric in $\mathbb{M}_n(R)$,

(7) If x is constant in R, then $[x;i,j]$ is constant in $\mathbb{M}_n(R)$,

(8) If x is distributive in R, then $[x;i,j]$ is distributive in $\mathbb{M}_n(R)$.

<u>Result</u> <u>2.4</u> (3.2 [6])

R is zero-symmetric if and only if $\mathbb{M}_n(R)$ is zero-symmetric.

Result 2.5 (3.3 [6])

If R is distributively-generated then so is $\mathbb{M}_n(R)$.

Result 2.6 (4.1 [6])

If L is a left ideal of R, then L^n is ~~a~~ an $\mathbb{M}_n(R)$-ideal of the $\mathbb{M}_n(R)$-module R^n.

Result 2.7 (4.2 [6])

If L is a left ideal of R, then

$$L^* := (L^n : R^n) := \{\mathfrak{X} \in \mathbb{M}_n(R) : \mathfrak{X}\alpha \in L^n \text{ for all } \alpha \in R^n\}$$

is a two-sided ideal of $\mathbb{M}_n(R)$.

Definition 2.8 (page 318 [6])

An ideal of $\mathbb{M}_n(R)$ is called full if it is of the form I^* for some ideal I of R.

Definition 2.9 (page 1 [14])

let I be an ideal of R. We define I^+ as an ideal of $\mathbb{M}_n(R)$ generated by the set $\{[a;i,j] : a \in I,\ 1 \leqslant i,\ j \leqslant n\}$.

Result 2.10 (1 [14])

$I^+ \subseteq I^*$ for any ideal I of R.

Result 2.11 (3.3 [13])

If G is a connected R-module, then G^n is a connected $\mathbb{M}_n(R)$-module.

Result 2.12 (3.4 [13])

If G is monogenic as an R-module, then G^n is monogenic as an $\mathbb{M}_n(R)$-module.

Result 2.13 (4.2 [13])

If I is an ideal in a zero-symmetric near-ring R, then

$$\mathbb{M}_n(R/I) \cong \mathbb{M}_n(R)/I^*.$$

Section B: Some Results on Matrix Near-Rings with Identity

Let R be a right near-ring with identity 1. The element of R^n with 1 in the i-th place and 0 elsewhere will be denoted by ε_i. The matrix units are the matrices $\mathcal{E}_{ij} := [1;i,j]$, $1 \leqslant i,j \leqslant n$.

$\mathbf{I} := \mathcal{E}_{11} + \mathcal{E}_{22} + \ldots + \mathcal{E}_{nn}$ is the identity element of $\mathbb{M}_n(R)$.

Result 2.14 (2.2 [6])

$\mathbb{M}_n(R)$ is the right near-ring with identity element $\mathbf{I}$.

Result 2.15 (2.3 [6])

If R is a ring with identity, then $\mathbb{M}_n(R)$ is isomorphic to the usual complete ring of n x n matrices over R.

Result 2.16 (3.1 [6])

(1) x is distributive in R if and only if $[x;i,j]$ is distributive in $\mathbb{M}_n(R)$.

(2) x is constant in R if and only if $[x;i,j]$ is constant in $\mathbb{M}_n(R)$.

Result 2.17 (3.4 [6])

If Δ is any non-empty subset of $\{1, 2, \ldots, n\}$, then

$\{\sum_{i\in\Delta} [x;i,i] : x \in R\}$ is a subnear-ring of $\mathbb{M}_n(R)$ which is isomorphic to R.

Result 2.18 (1) (4.3 [6])

If I_1 and I_2 are ideals in R and $I_1 \neq I_2$, then $I_1^* \neq I_2^*$.

(2) (2 [14])

The mapping $I \to I^+$ is an injection.

Definition 2.19 (page 317 [6])

If $\mathcal{I}$ is an ideal in $\mathbb{M}_n(R)$, then

$\mathcal{I}_*(R) := \{a \in R\colon a = \pi_j \mathcal{A}\alpha$, for some $\mathcal{A} \in \mathcal{I}$, $\alpha \in R^n$ and j, $1 \leqslant j \leqslant n\}$

Remark: R is, henceforth, assumed to be a zero-symmetric near-ring, for this section.

Result 2.20 (4.4 [6])

Let $\mathcal{I}$ be an ideal in $\mathbb{M}_n(R)$. Then $a \in \mathcal{I}_*$ if and only if

$[a;1,1] \in \mathcal{I}$.

Result 2.21 (4.5 [6])

Let $\mathcal{I}$ be an ideal in $\mathbb{M}_n(R)$. Then $a \in \mathcal{I}_*$ if and only if

$[a;i,j] \in \mathcal{I}$, where $1 \leqslant i,j \leqslant n$.

Result 2.22 (4.6 [6])

If $\mathcal{I}$ is an ideal in $\mathbb{M}_n(R)$, then $\mathcal{I}_*$ is an ideal in R.

Result 2.23 (4.7 [6])

If $\mathcal{I}$ is an ideal in $\mathbb{M}_n(R)$ and I is an ideal in R, then

(1) $(\mathcal{I}_*)^* \supseteq \mathcal{I}$

(2) $(I^*)_* = I$

(3) $((\mathcal{I}_*)^*)_* = \mathcal{I}_*$.

Result 2.24 (4.8 [6])

There is a bijection between the set of ideals of R and the set of full ideals of $\mathbb{M}_n(R)$ given by $I \to I^*$ and $\mathcal{I} \to \mathcal{I}_*$ such that $(I^*)_* = I$ and $(\mathcal{I}_*)^* = \mathcal{I}$, for any ideal I of R and a full ideal $\mathcal{I}$ of $\mathbb{M}_n(R)$.

Result 2.25 (3 [14])

If $\mathcal{I}$ is an ideal in $\mathbb{M}_n(R)$, then $(\mathcal{I}_*)^+ \subseteq \mathcal{I} \subseteq (\mathcal{I}_*)^*$.

Result 2.26 (Example 4 [14])

Let $R = Z_0[x]$, the zero-symmetric polynomial near-ring in one indeterminate over the integers (see page 220 [9]). Consider the ideal $I := (2)[x]$. Then in $\mathbb{M}_2(R)$, $I^+ \subsetneq I^*$.

Result 2.27 (3.7 [13])

If G is a connected R-module. Then any $\mathbb{M}_n(R)$-ideal (submodule) of G^n is of the form I^n, where I is an R-ideal (submodule) of G.

Result 2.28 (3.8 [13])

If G is a connected R-module, then G is simple (R-simple) if and only if G^n is simple ($\mathbb{M}_n(R)$-simple).

Result 2.29 (3.9 [13])

If R is ν-primitive on G then $\mathbb{M}_n(R)$ is ν-primitive on G^n, where $\nu = 0, 2$.

Result 2.30 (3.10 [13])

R is 2-primitive if and only if $\mathbb{M}_n(R)$ is 2-primitive.

Result 2.31 (4.3 [13])

An ideal $\mathcal{I}$ in $\mathbb{M}_n(R)$ is 2-primitive if and only if $\mathcal{I} = I^*$ for some 2-primitive ideal I of R.

Result 2.32 (4.4 [13])

$$J_2(\mathbb{M}_n(R)) = (J_2(R))^*.$$

Result 2.33 (12 [14])

Let R be a d.g. near-ring with identity, and I be an ideal in R. If $I \supseteq \delta_1(R)$, the commutator subgroup of $(R,+)$, then I^* in $\mathbb{M}_2(R)$ consists of all and only those matrices of the form

$$[a_1;1,1] + [b_1;1,2] + \ldots + [a_m;1,1] + [b_m;1,2]$$
$$+ [c_1;2,1] + [d_1;2,2] + \ldots + [c_q;2,1] + [d_q;2,2]$$

where $\sum_{t=1}^{m} a_t$, $\sum_{t=1}^{m} b_t$, $\sum_{i=1}^{q} c_i$, $\sum_{i=1}^{q} d_i \in I$.

Note: This result can be extended to $\mathbb{M}_n(R)$, $n > 2$.

Section C: Some Results due to J. H. Meyer

Definition 2.34 (page 24 [8])

Let R be a near-ring, and L be a left ideal of R. R/L may be considered as a left R-module, where $x(y + L) := xy + L$ for all $x, y \in R$. Similarly $(R/L)^n$ can be defined as an $\mathbb{M}_n(R)$-module as follows: Let $\alpha := \langle x_1 + L, \ldots, x_n + L\rangle \in (R/L)^n$. Then for $x \in R$ and

$1 \leqslant i, j \leqslant n$, $[x;i,j]\alpha := \langle 0 + L, .., xx_j + L, .., 0 + L\rangle$, with $xx_j + L$ in the i-th position of the vector.

Let $\mathcal{X}$, $\mathcal{Y} \in \mathbb{M}_n(R)$. Suppose $\mathcal{X}\beta$ and $\mathcal{Y}\beta$ are defined for all $\beta \in (R/L)^n$. Then $(\mathcal{X} + \mathcal{Y})\beta$ is defined by $\mathcal{X}\beta + \mathcal{Y}\beta$ and $(\mathcal{X}\mathcal{Y})\beta$ by $\mathcal{X}(\mathcal{Y}\beta)$. This definition is well-defined because of the following result.

<u>Result</u> <u>2.35</u> (1.28 [8])

Let R be a near-ring and L be a left ideal of R. If $\mathcal{X} \in \mathbb{M}_n(R)$ and $\langle x_1, \ldots, x_n\rangle \in R^n$ with $\mathcal{X}\langle x_1, \ldots, x_n\rangle = \langle y_1, \ldots, y_n\rangle$, then $\mathcal{X}\langle x_1 + L, \ldots, x_n + L\rangle = \langle y_1 + L, \ldots, y_n + L\rangle$.

Proof: We use induction on the weight, $w(\mathcal{X})$, of $\mathcal{X}$. If $w(\mathcal{X}) = 1$ and $\mathcal{X} = [x;i.j]$, for some $x \in R$ and $1 \leqslant i, j \leqslant n$, then $[x;i,j]\ \langle x_1,\ldots,x_n\rangle = \langle 0,..,xx_j,..,0\rangle$, with xx_j in the i-th place.

Now $[x;i,j]\langle x_1 + L,..,x + L_n\rangle = \langle 0 + L,..,xx_j + L,..,0 + L\rangle$, with $xx_j + L$ in the i-th position of the vector, by definition 2.34.

We next assume that the result is true for all $\mathcal{Y} \in \mathbb{M}_n(R)$ with $w(\mathcal{Y}) < m$, $m \in \mathbb{N}$, $m > 1$.

If $w(\mathcal{X}) = m$, then there are two possibilities:

$\mathcal{X} = \mathcal{X}_1 + \mathcal{X}_2$ or $\mathcal{X} = \mathcal{X}_1\mathcal{X}_2$ where $w(\mathcal{X}_1)$, $w(\mathcal{X}_2) < m$.

(1) If $\mathcal{X} = \mathcal{X}_1 + \mathcal{X}_2$, $\mathcal{X}_1\langle x_1,\ldots,x_n\rangle = \langle y_1,\ldots,y_n\rangle$ and $\mathcal{X}_2\langle x_1,\ldots,x_n\rangle = \langle z_1,\ldots,z_n\rangle$, then indeed $\mathcal{X}\langle x_1,\ldots,x_n\rangle = \langle y_1 + z_1,\ldots,y_n + z_n\rangle$.

Now $\mathfrak{X}<x_1 + L,\dots,x_n + L> = (\mathfrak{X}_1 + \mathfrak{X}_2)<x_1 + L,\dots,x_n + L>$

$$= \mathfrak{X}_1<x_1+ L,\dots,x_n+ L> + \mathfrak{X}_2<x_1+ L,\dots,x_n+ L>$$

$$= <y_1+ L,\dots,y_n+ L> + <z_1+ L,\dots,z_n+ L>$$

by the induction hypothesis,

$$= <y_1+ z_1+ L,\dots,y_n+ z_n+ L>.$$

(2) If $\mathfrak{X} = \mathfrak{X}_1 X_2$, $\mathfrak{X}_2<x_1,\dots x_n> = <z_1,\dots,z_n>$ and

$\mathfrak{X}_1<z_1,\dots,z_n> = <w_1,\dots,w_n>$, then indeed $\mathfrak{X}<x_1,\dots,x_n> = <w_1,\dots,w_n>$.

Now $\mathfrak{X}<x_1+ L,\dots,x_n+ L> = (\mathfrak{X}_1\mathfrak{X}_2)<x_1+ L,\dots,x_n+ L>$

$$= \mathfrak{X}_1(\mathfrak{X}_2<x_1+ L,\dots,x_n+ L>)$$

$$= \mathfrak{X}_1<z_1+ L,\dots,z_n+ L>$$

by the induction hypothesis,

$$= <w_1+ L,\dots,w_n+ L>$$

by the induction hypothesis.

The result now follows by induction.

<u>Result 2.36</u> (1.29 [8])

Let R be a near-ring and L be a left ideal of R. Then

$R^n/L^n \cong (R/L)^n$ as $\mathbb{M}_n(R)$-modules.

Proof: Define the function $\theta : R^n \to (R/L)^n$ by

$<x_1,\dots x_n>\theta = <x_1+L,\dots,x_n+L>$, for every $<x_1,\dots,x_n> \in R^n$.

It can be checked easily that θ is an $\mathbb{M}_n(R)$-epimorphism, as

$(\mathfrak{X}\alpha)\theta = \mathfrak{X}(\alpha\theta)$, for all $\mathfrak{X} \in \mathbb{M}_n(R)$ and $\alpha \in R^n$, by result 2.35.

Moreover, $\mathrm{Ker}\theta = \{<a_1,\ldots,a_n> \in R^n : a_i \in L,\ i = 1,2,\ldots,n\}$

$= L^n$.

Hence $R^n/L^n = R^n/\mathrm{Ker}\theta \cong (R/L)^n$ as $\mathbb{M}_n(R)$-modules.

<u>Remarks</u> <u>2.37</u> (1) (page 25 [8])

If L is a left ideal of R and χ is any non-empty subset of R^n, then

$(L^n:\chi) := \{\mathcal{A} \in \mathbb{M}_n(R) : \mathcal{A}\chi \subseteq L^n\}$ is a left ideal of $\mathbb{M}_n(R)$, by result 1.42 (Pilz [9]).

In particular, $(L^n:\alpha) := (L^n:\{\alpha\}) := \{\mathcal{A} \in \mathbb{M}_n(R) : \mathcal{A}\alpha \in L^n\}$

is a left ideal of $\mathbb{M}_n(R)$ for any $\alpha \in R^n$.

(2) (page 26 [8])

If R is a zero-symmetric near-ring with identity, then

$(L^n:\alpha) = \mathbb{M}_n(R)$ if and only if $\alpha \in L^n$.

<u>Result</u> <u>2.38</u> (1.30 [8])

Let R be a zero-symmetric near-ring with identity. If L is a strictly maximal left ideal of R and $\alpha \in R^n - L^n$, then $(L^n:\alpha)$ is a strictly maximal left ideal of $\mathbb{M}_n(R)$.

Proof: Let $\psi : R^n \to R^n/L^n$ be the canonical $\mathbb{M}_n(R)$-epimorphism. Define the function $\varphi : \mathbb{M}_n(R) \to R^n$ by $\mathcal{X}\varphi = \mathcal{X}\alpha$ for every $\mathcal{X} \in \mathbb{M}_n(R)$. φ is indeed an $\mathbb{M}_n(R)$-homomorphism as $(\mathcal{X}\mathcal{Y})\varphi = \mathcal{X}(\mathcal{Y}\varphi)$ for all $\mathcal{X}, \mathcal{Y} \in \mathbb{M}_n(R)$. Hence $\varphi\psi : \mathbb{M}_n(R) \to R^n/L^n$ is an $\mathbb{M}_n(R)$-homomorphism. Since L is a strictly maximal ideal in R, so R/L is an R-simple R-module, by result 1.30, so $(R/L)^n$ is an $\mathbb{M}_n(R)$-simple $\mathbb{M}_n(R)$-module, by result 2.28. Therefore R^n/L^n is an $\mathbb{M}_n(R)$-simple $\mathbb{M}_n(R)$-module, as $(R/L)^n \cong R^n/L^n$ as $\mathbb{M}_n(R)$-module, by result 2.36. It follows that $Im(\varphi\psi) = 0$ or R^n/L^n.

But we have
$$\begin{aligned} I(\varphi\psi) &= (I\varphi)\psi, \text{ where } I \text{ is the identity of } \mathbb{M}_n(R). \\ &= (I\alpha)\psi \\ &= \alpha\psi \\ &= \alpha + L^n \\ &\neq 0, \text{ as } \alpha \notin L^n. \end{aligned}$$

Thus $Im(\varphi\psi) = R^n/L^n$. This ensures that $\varphi\psi$ is an $\mathbb{M}_n(R)$-epimorphism. Furthermore,

$$\begin{aligned} Ker(\varphi\psi) &= \{\mathcal{A} \in \mathbb{M}_n(R) : (\mathcal{A}\varphi)\psi = L^n/L^n = \{0\}\} \\ &= \{\mathcal{A} \in \mathbb{M}_n(R) : \mathcal{A}\alpha \in L^n \} \\ &= (L^n : \alpha). \end{aligned}$$

This implies that $\mathbb{M}_n(R)/(L^n:\alpha)$ is $\mathbb{M}_n(R)$-isomorphic to R^n/L^n. Since R^n/L^n is an $\mathbb{M}_n(R)$-simple as an $\mathbb{M}_n(R)$-module, so $\mathbb{M}_n(R)/(L^n:\alpha)$ is also $\mathbb{M}_n(R)$-simple as an $\mathbb{M}_n(R)$-module. Consequently, $(L^n:\alpha)$ is a maximal left ideal of $\mathbb{M}_n(R)$ and is not properly contained in a proper left $\mathbb{M}_n(R)$-subgroup of $\mathbb{M}_n(R)$.

<u>Result</u> <u>2.39</u> (2.1 [8])

The elements of a zero-symmetric near-ring R may be viewed as an $\mathbb{M}_n(R)$-endomorphism of R^n where the action of R on R^n is considered as multiplication on the right

$<x_1,\ldots,x_n>r = <x_1r,\ldots,x_nr>$

Proof: Let α, $\beta \in R^n$ and $r \in R$. Then

$(\alpha + \beta)r = \alpha r + \beta r$, by right distributivity.

We need to show that $\mathcal{X}(\alpha r) = (\mathcal{X}\alpha)r$, for arbitrary $\mathcal{X} \in \mathbb{M}_n(R)$, $\alpha \in R^n$ and $r \in R$.

We use induction on the weight, $w(\mathcal{X})$, of $\mathcal{X}$.

If $w(\mathcal{X}) = 1$ and $\mathcal{X} = [x;i,j]$ for some $x \in R$ and $1 \leqslant i,\ j \leqslant n$, then

$([x;i,j]<x_1,\ldots,x_n>)r$

$= <0,\ldots,xx_j,\ldots,0>r$

with xx_j in the i-th position of the vector,

$= <0,\ldots,xx_jr,\ldots,0>$, as R is zero-symmetric,

$= [x;i,j]<x_1r,\ldots,x_jr,\ldots,x_nr>$.

We next assume that the result is true for all $\mathcal{Y} \in \mathbb{M}_n(R)$ with $w(\mathcal{Y}) < m$, $m \in \mathbb{N}$, $m > 1$. If $w(\mathcal{X}) = m$, then there are two possibilities:

$\mathcal{X} = \mathcal{X}_1 + \mathcal{X}_2$ or $\mathcal{X} = \mathcal{X}_1\mathcal{X}_2$ where $w(\mathcal{X}_1)$, $w(\mathcal{X}_2) < m$.

(1) If $\mathcal{X} = \mathcal{X}_1 + \mathcal{X}_2$, then

$((\mathcal{X}_1 + \mathcal{X}_2)<x_1,\ldots,x_n>)r = (\mathcal{X}_1<x_1,\ldots,x_n> + \mathcal{X}_2<x_1,\ldots,x_n>)r$

$= (<y_1,\ldots,y_n> + <z_1,\ldots,z_n>)r$

where $y_1,\ldots,y_n$, $z_1,\ldots,z_n \in R$,

$= <y_1 + z_1,\ldots,y_n + z_n>r$

$$= \langle (y_1 + z_1)r, \ldots, (y_n + z_n)r \rangle$$

$$= \langle y_1 r + z_1 r, \ldots, y_n r + z_n r \rangle$$

by right distributivity,

$$= \langle y_1 r, \ldots, y_n r \rangle + \langle z_1 r, \ldots, z_n r \rangle$$

$$= \langle y_1, \ldots, y_n \rangle r + \langle z_1, \ldots, z_n \rangle r$$

$$= (\mathfrak{X}_1 \langle x_1, \ldots, x_n \rangle) r + (\mathfrak{X}_2 \langle x_1, \ldots, x_n \rangle) r$$

$$= \mathfrak{X}_1 (\langle x_1, \ldots, x_n \rangle r) + \mathfrak{X}_2 (\langle x_1, \ldots, x_n \rangle r)$$

by the induction hypothesis,

$$= (\mathfrak{X}_1 + \mathfrak{X}_2) \langle x_1 r, \ldots, x_n r \rangle.$$

(2) If $\mathfrak{X} = \mathfrak{X}_1 \mathfrak{X}_2$, then

$$((\mathfrak{X}_1 \mathfrak{X}_2) \langle x_1, \ldots, x_n \rangle) r = (\mathfrak{X}_1 (\mathfrak{X}_2 \langle x_1, \ldots, x_n \rangle)) r$$

$$= (\mathfrak{X}_1 \langle z_1, \ldots, z_n \rangle) r$$

$$= \mathfrak{X}_1 (\langle z_1, \ldots, z_n \rangle r)$$

by the induction hypothesis,

$$= \mathfrak{X}_1 \langle z_1 r, \ldots, z_n r \rangle$$

$$= \mathfrak{X}_1 (\mathfrak{X}_2 \langle x_1 r, \ldots, x_n r \rangle)$$

$$= \mathfrak{X}_1 (\mathfrak{X}_2 \langle x_1, \ldots, x_n \rangle r)$$

$$= \mathfrak{X}_1 \mathfrak{X}_2 (\langle x_1, \ldots, x_n \rangle r)$$

by the induction hypothesis,

$$= (\mathfrak{X}_1 \mathfrak{X}_2) \langle x_1 r, \ldots, x_n r \rangle.$$

The result now follows by induction.

CHAPTER 3

ON MATRIX NEAR-RINGS

We may briefly indicate the contents of the present chapter by saying that the structure of a matrix near-ring $\mathbb{M}_n(R)$ and the behaviour of its ideals I^+ and I^*, where I is an ideal of R, are studied.

Recall that throughout this presentation we will, in general, be dealing with near-rings not necessarily with identity (unless otherwise specified).

We start with a couple of results which show the similarities to the ring case.

It is well known that, if $n > 1$, a matrix ring $M_n(R)$ has proper divisors of zero even though R has none (see page 154 McCoy [3]), and if R is a ring with identity then a sum of distinct matrix units of the form E_{kk}, $1 \leqslant k \leqslant n$, is an idempotent in the complete matrix ring $M_n(R)$ (see page 20 McCoy [4]). We show that these results are also true in the near-ring case.

We need a result which appeared in the proof of lemma 2.1 Van Der Walt [13]. It is of such importance that we prove it explicitly.

Lemma 3.1 : Let R be a near-ring with identity and $x, y \in R$. If $[x;i,j] = [y;i,j]$, then $x = y$ where $1 \leq i,j \leq n$.

Proof :

Let $[x;i,j] = [y;i,j]$. Then $[x;i,j]\varepsilon_j = [y;i,j]\varepsilon_j$ where

$\varepsilon_j = \langle 0,..,1,..,0 \rangle$ with 1 in the j-th place.

So $\langle 0,..,x,..,0 \rangle = \langle 0,..,y,..,0 \rangle$ with x and y in the i-th place.

Hence $x = y$.

Theorem 3.2 : Let R be a zero-symmetric near-ring with identity. If $n > 1$, then $\mathbb{M}_n(R)$ can not be integral.

Proof: Presuming the theorem to be false, we choose any two non-zero elements, say x and y, of R.

Since $[x;i,j] \cdot [y;k,l] = \mathbf{0}$, if $j \neq k$, by result 2.3, as $x0 = 0$, therefore either $[x;i,j] = \mathbf{0}$ or $[y;k,l] = \mathbf{0}$, as $\mathbb{M}_n(R)$ is an integral near-ring. In other words, $[x;i,j] = [0;i,j]$ or $[y;k,l] = [0;k,l]$.

Hence $x = 0$ or $y = 0$, by lemma 3.1. This leads to a contradiction.

Theorem 3.3: Let R be a near-ring with identity. A sum of distinct matrix units of the form $\mathcal{E}_{kk}$, $1 \leqslant k \leqslant n$, is an idempotent in $\mathbb{M}_n(R)$.

Proof: $\mathcal{E}_{ii}$, for $i = 1, 2, \ldots, n$, is an idempotent as

$\mathcal{E}_{ii}\mathcal{E}_{ii} = [1;i,i] \cdot [1;i,i] = [1;i,i] = \mathcal{E}_{ii}$, by result 2.3 (3).

We aim to show that $\mathcal{E}_{11} + \ldots + \mathcal{E}_{ll}$ is an idempotent, where

$1 \leqslant l \leqslant n$.

$(\mathcal{E}_{11}+..+ \mathcal{E}_{ll})(\mathcal{E}_{11}+..+ \mathcal{E}_{ll}) = \mathcal{E}_{11}(\mathcal{E}_{11}+..+ \mathcal{E}_{ll}) +..+ \mathcal{E}_{ll}(\mathcal{E}_{11}+..+ \mathcal{E}_{ll})$,

by right distributive law,

$= \mathcal{E}_{11}+\ldots+ \mathcal{E}_{ll}$, by result 2.3 (5).

This completes the proof.

The following result is an extention of result 2.3 (7).

Lemma 3.4: If R is a constant near-ring then so is $\mathbb{M}_n(R)$.

Proof: Let $\mathcal{X} \in \mathbb{M}_n(R)$. We need to show that $\mathcal{X}\mathbf{0} = \mathcal{X}$, where $\mathbf{0}$ is the zero matrix.

We use induction on the weight, $w(\mathcal{X})$, of $\mathcal{X}$.

If $w(\mathcal{X}) = 1$ and $\mathcal{X} = [x;i,j]$, then $[x;i,j]$ is constant, by result 2.3 (7).

Assume that $\mathcal{Y}\mathbf{0} = \mathcal{Y}$ whenever $w(\mathcal{Y}) < m$, $m \in \mathbb{N}$, $m \geqslant 2$. If $w(\mathcal{X}) = m$, then there are two possibilities: $\mathcal{X} = \mathcal{X}_1 + \mathcal{X}_2$ or $\mathcal{X} = \mathcal{X}_1\mathcal{X}_2$, where $w(\mathcal{X}_1)$, $w(\mathcal{X}_2) < m$.

(1) If $\mathcal{X} = \mathcal{X}_1 + \mathcal{X}_2$, then $\mathcal{X}\mathbf{0} = (\mathcal{X}_1 + \mathcal{X}_2)\mathbf{0}$

$= \mathcal{X}_1\mathbf{0} + \mathcal{X}_2\mathbf{0}$, by right distributive law,

$= \mathcal{X}_1 + \mathcal{X}_2$, by the induction hypothesis,

$= \mathcal{X}$.

(2) If $\mathcal{X} = \mathcal{X}_1\mathcal{X}_2$, then $\mathcal{X}\mathbf{0} = (\mathcal{X}_1\mathcal{X}_2)\mathbf{0}$

$= \mathcal{X}_1(\mathcal{X}_2\mathbf{0})$

$= \mathcal{X}_1\mathcal{X}_2$, by the induction hypothesis,

$= \mathcal{X}$.

The result now follows by induction.

We now come to establish the results which supply strong links between the additive structure of R and that of $\mathbb{M}_n(R)$.

For one of them we give two different proofs.

Lemma 3.5: Let R be a near-ring. If $(R,+) \in V$, a variety of additive groups, then $(\mathbb{M}_n(R),+) \in V$.

Proof 1: Recall that we define a matrix near-ring $\mathbb{M}_n(R)$ over an arbitrary near-ring R as a subnear-ring of $M(R^n)$ (see definition 2.1 and the remark following it). This means that $(R^n,+)$ is a faithful $\mathbb{M}_n(R)$-module. Now if $(R,+) \in V$, then $(R^n,+) \in V$, by result 1.12. These two facts together with result 1.40 force that $(\mathbb{M}_n(R),+) \in V$.

Proof 2: Let $w(v_1,\ldots,v_p)$ be a law of V, that is,

$$w(v_1,\ldots,v_p) = \pm v_{1_1} \pm \ldots\ldots \pm v_{1_t} = 0$$

where $\{1_1,\ldots,1_t\} \subseteq \{1,\ldots,p\}$.

If $\mathcal{X}_1,\ldots,\mathcal{X}_p \in \mathbb{M}_n(R)$, we must show that $w(\mathcal{X}_1,\ldots,\mathcal{X}_p) = \mathbf{0}$.

Since $\mathcal{X}_1\alpha,\ldots,\mathcal{X}_p\alpha \in R^n$, for all $\alpha \in R^n$, and $(R^n,+) \in V$, by result 1.12, so $w(\mathcal{X}_1\alpha,\ldots\mathcal{X}_p\alpha) = \underline{0}$. Therefore $w(\mathcal{X}_1,\ldots,\mathcal{X}_p)\alpha = \underline{0}$, by lemma 3.21, which is a rewording of result 1.39. Thus definition 1.25 forces that $w(\mathcal{X}_1,\ldots,\mathcal{X}_p) \in \mathrm{Ann}_{\mathbb{M}_n(R)}(R^n,+) = 0$, as $(R^n,+)$ is a faithful $\mathbb{M}_n(R)$-module (see definition 2.1).

Therefore the law $w(\mathcal{X}_1,\ldots,\mathcal{X}_p)$ holds in $(\mathbb{M}_n(R),+)$. This is true for all the laws of V. Thus $(\mathbb{M}_n(R),+) \in V$.

Theorem 3.6: Let R be a near-ring with identity. Then $(R,+) \in V$ if and only if $(\mathbb{M}_n(R),+) \in V$.

Proof: The necessary condition has been proved in lemma 3.5. Conversely, let $(\mathbb{M}_n(R),+) \in V$. Since $\mathbb{M}_n(R)$ has a subnear-ring isomorphic to R (see result 2.17) and every variety is closed with respect to forming subgroups and images (see result 1.12), therefore $(R,+) \in V$.

Remark: The converse in theorem 3.6 is not true, in general. The following example will justify our claim.

<u>Example 3.7</u>: Let R be a distributive near-ring which is not a ring (obviously, R is a non abelian near-rng). Assume that the converse in theorem 3.6 is true, in general. Since R is distributive, therefore $(\mathbb{M}_n(R),+)$ is abelian, by theorem 4.4, which will be proved in chapter 4. Thus $(R,+)$ is abelian, by assumption. This is a contradiction.

The following series of results is an immediate consequence of lemma 3.5 and theorem 3.6, and will be useful in the sequel.

<u>Corollary 3.8</u>: Let R be a near-ring such that $(R,+)$ is soluble. Then $(\mathbb{M}_n(R),+)$ is soluble.

<u>Corollary 3.9</u>: Let R be a near-ring with identity. Then $(R,+)$ is soluble if and only if $(\mathbb{M}_n(R),+)$ is soluble.

<u>Corollary 3.10</u>: Let R be a near-ring such that $(R,+)$ is nilpotent. Then $(\mathbb{M}_n(R),+)$ is nilpotent.

<u>Corollary 3.11</u>: Let R be a near-ring with identity. Then $(R,+)$ is nilpotent if and only if $(\mathbb{M}_n(R),+)$ is nilpotent.

<u>Corollary 3.12</u>: Let R be a near-ring such that (R,+) is $\bar{c}$-nilpotent. Then $(\mathbb{M}_n(R),+)$ is $\bar{c}$-nilpotent.

<u>Corollary 3.13</u>: Let R be a near-ring with identity. Then (R,+) is $\bar{c}$-nilpotent if and only if $(\mathbb{M}_n(R),+)$ is $\bar{c}$-nilpotent.

Recall that J. D. P. Meldrum proved that, if G is a faithful R-module and $G \in V$, a variety of additive groups, then $(R,+) \in V$ (see result 1.40). We wish to extend this result.

<u>Theorem 3.14:</u> Let G be a connected R-module. If $(R,+) \in V$, then $G \in V$.

Proof: Let $w(x_1,\ldots,x_p)$ be a law in V. Suppose that $g_1,\ldots,g_p \in G$. We must show that $w(g_1,\ldots,g_p) = 0_G$, where 0_G is the additive identity of G. Since G is a connected R-module, so for each $g_1,\ldots,g_p \in G$, there exist g in G and $r_1,\ldots,r_p$ in R such that $g_1 = r_1g,\ldots, g_p = r_pg$, by result 1.34.

Now $w(g_1,\ldots,g_p) = w(r_1g,\ldots,r_pg) = w(r_1,\ldots,r_p)g$, by result 1.39. Now since $(R,+) \in V$, so $w(r_1,\ldots,r_p) = 0_R$, where 0_R is the additive identity of R. Therefore $w(g_1,\ldots,g_p) = 0_Rg = 0_G$, by result 1.24. This is true for all the laws of V. Hence $G \in V$.

<u>Theorem 3.15</u>: Let (R,+) be a connected R-module. Then $(R,+) \in V$ if and only if $(\mathbb{M}_n(R),+) \in V$.

Proof: The necessary condition follows from lemma 3.5, so only the converse needs proof. It is known that if G is a connected R-module, then G^n is a connected $\mathbb{M}_n(R)$-module (see result 2.11).

We have $(R^n,+) \in V$, by theorem 3.14, as $(R^n,+)$ is a connected $\mathbb{M}_n(R)$-module and $(\mathbb{M}_n(R),+) \in V$. This forces $(R,+)$ to be in V.

There are some immediate consequences of theorem 3.15.

<u>Corollary</u> <u>3.16</u>: Let $(R,+)$ be a connected R-module. Then

(1) $(R,+)$ is abelian if and only if $(\mathbb{M}_n(R),+)$ is abelian.

(2) $(R,+)$ is nilpotent if and only if $(\mathbb{M}_n(R),+)$ is nilpotent.

(3) $(R,+)$ is $\bar{c}$-nilpotent if and only if $(\mathbb{M}_n(R),+)$ is $\bar{c}$-nilpotent.

(4) $(R,+)$ is soluble if and only if $(\mathbb{M}_n(R),+)$ is soluble.

Theorem 3.14 and result 1.40 together give us the following result.

<u>Theorem</u> <u>3.17</u>: Let G be a faithful connected R-module. Then $G \in V$ if and only if $(R,+) \in V$.

Remark: We have the analogous results of theorem 3.14 to theorem 3.17 for a monogenic R-module, as every monogenic R-module is connected.

<u>Corollary</u> <u>3.18</u>: Let R be a ν-primitive near-ring on a R-module G, where $\nu = 0, 1, 2$. Then $G \in V$ if and only if $(R,+) \in V$.

Proof: The result follows immediately from the previous remark and theorem 3.17, as G is a faithful monogenic R-module.

Before going on to work on the relationship between ideals of R and those of $\mathbb{M}_n(R)$, it may be worth noting that the maps $I \to I^*$ and $I \to I^+$, which are injections of the set of all ideals in a near-ring

R with identity into the set of all ideals in $\mathbb{M}_n(R)$ (see result 2.18), need not be so, in general.

The following examples will prove our claim.

Example 3.19: Let R be a zero-symmetric near-ring without identity and I be a non-trivial proper ideal of R such that

$xy \in I$ for all $x, y \in R$. We aim to show that $\mathbb{M}_n(R) = I^*$.

Let $\mathcal{X} \in \mathbb{M}_n(R)$. We use induction on the weight, $w(\mathcal{X})$, of $\mathcal{X}$.

Let $w(\mathcal{X}) = 1$ and $\mathcal{X} = [x;i,j]$, where $x \in R$ and $1 \leqslant i,j \leqslant n$.

If $\alpha = \langle r_1,\ldots,r_n\rangle \in R^n$, then $[x;i,j]\alpha = \langle 0,\ldots,xr_j,\ldots,0\rangle$, with xr_j in the i-th position of the vector. Now $[x;i,j]\alpha \in I^n$, as $xr_j \in I$.

So $[x;i,j] \in I^*$.

Next we assume that $\mathcal{Y}\alpha \in I^n$ whenever $w(\mathcal{Y}) < m$, $m \in \mathbb{N}$, $m \geqslant 2$.

If $w(\mathcal{X}) = m$, then there are two possibilities: $\mathcal{X} = \mathcal{X}_1 + \mathcal{X}_2$ or $\mathcal{X} = \mathcal{X}_1\mathcal{X}_2$, where $w(\mathcal{X}_1)$, $w(\mathcal{X}_2) < m$.

The first case can be checked easily. We show the second case.

$\mathcal{X}\alpha = (\mathcal{X}_1\mathcal{X}_2)\alpha = \mathcal{X}_1(\mathcal{X}_2\alpha) = \mathcal{X}_1\beta$, where $\beta = \mathcal{X}_2\alpha \in I^n$, by the induction hypothesis. Now since $\mathbb{M}_n(R)\ I^n \subseteq I^n$, by result 1.31, as $\mathbb{M}_n(R)$ is zero-symmetric and I^n is an $\mathbb{M}_n(R)$-ideal of R^n (see results 2.4 and 2.6), therefore $\mathcal{X}\alpha = \mathcal{X}_1\beta \in I^n$. Hence $\mathcal{X} \in I^*$.

This implies that $\mathbb{M}_n(R) = I^*$, that is, $R^* = I^*$, as $\mathbb{M}_n(R) = R^*$.

Hence $(\ )^*$ is not an injection.

Example 3.20: Let R be a distributive near-ring which is not a ring. (Obviously, R is a non-abelian near-rng.) We have

$(R,R)^+ \subseteq (R^+,R^+)$, by lemma 4.17, which will be proved in chapter 4.

Since $R^+ = \mathbb{M}_n(R)$, so $(R,R)^+ \subseteq (\mathbb{M}_n(R),\mathbb{M}_n(R)) = \mathbf{0}$, as $(\mathbb{M}_n(R),+)$

is abelian , by theorem 4.4, which will be proved in chapter 4.

Therefore $(R,R)^+ = \{0\}^+$, as $\{0\}^+ = \mathbf{0}$. Now if $(\ )^+$ is an injection, then $(R,R) = \{0\}$. This contradicts the assumption.

Hence $(\ )^+$ is not an injection.

Our next series of results takes lemma 3.5 to corollary 3.13 further. We first state a lemma which is a rewording of result 1.39.

Lemma 3.21: Let $w(v_1,\dots v_p)$ be a word in p variables $v_1,\dots,v_p$.

Then $w(\mathscr{X}_1,\dots,\mathscr{X}_p)\alpha = w(\mathscr{X}_1\alpha,\dots,\mathscr{X}_p\alpha)$, where $\mathscr{X}_1,\dots,\mathscr{X}_p \in \mathbb{M}_n(R)$ and $\alpha \in R^n$.

Theorem 3.22: Let R be a near-ring and I be an ideal of R. If $(I,+) \in V$, then $(I^*,+) \in V$.

Proof: Let $w(v_1,\dots,v_p) = \mathbf{0}$ be a law of V, that is,

$$w(v_1,\dots,v_p) = \pm v_{l_1} \pm \dots \pm v_{l_t} = 0$$

where $\{l_1,\dots,l_t\} \subseteq \{1,\dots,p\}$.

If $\mathscr{A}_1,\dots,\mathscr{A}_p \in I^*$, we must show that $w(\mathscr{A}_1,\dots,\mathscr{A}_p) = \mathbf{0}$.

Since $\mathscr{A}_1\alpha,\dots,\mathscr{A}_p\alpha \in I^n$, for all $\alpha \in R^n$, by definition of I^*, and $(I^n,+) \in V$, by result 1.12, so $w(\mathscr{A}_1\alpha,\dots,\mathscr{A}_p\alpha) = \underline{0}$. That is,

$w(\mathscr{A}_1,\dots,\mathscr{A}_p)\alpha = \underline{0}$, by lemma 3.21. Therefore

$w(\mathcal{A}_1,\ldots,\mathcal{A}_p) \in \mathrm{Ann}_{\mathbb{M}_n(R)}(R^n,+) = 0$, as $(R^n,+)$ is a faithful $\mathbb{M}_n(R)$-module.

Hence the law $w(\mathcal{A}_1,\ldots,\mathcal{A}_p) = 0$ holds in $(I^*,+)$. This is true for all laws of V. Thus $(I^*,+) \in V$.

The following lemma enables us to prove the converse of this result.

<u>Lemma</u> <u>3.23</u>: $w([a_1;i,j],\ldots,[a_p;i,j]) = [w(a_1,\ldots,a_p);i,j]$ where $a_1,\ldots,a_p \in R$ and $1 \leqslant i,j \leqslant n$.

Proof: We use induction on the length q of the word $w(v_1,\ldots,v_p)$. If $q = 1$, then $w(v_1,\ldots,v_p) = \pm\, v_t$, for some v_t. Therefore

$$\begin{aligned} w([a_1;i,j],\ldots,[a_p;i,j]) &= \pm\,[a_t;i,j] \\ &= [\pm\, a_t;i,j], \text{ by } 2.3(4), \\ &= [w(a_1,\ldots,a_p);i,j]. \end{aligned}$$

Hence the result is true for $q = 1$. Next we assume that the result holds for words of length q and let the word $w(v_1,\ldots,v_p)$ have length $q+1$. Then $w(v_1,\ldots,v_p) = w_1(v_1,\ldots,v_p) \pm v_t$, for some v_t and some word $w_1(v_1,\ldots,v_p)$ of length q. Now

$$w([a_1;i,j],\ldots,[a_p;i,j]) = w_1([a_1;i,j],\ldots,[a_p;i,j]) \pm [a_t;i,j]$$

$$= [w_1(a_1,\ldots,a_p);i,j] \pm [a_t;i,j]$$

by the induction hypothesis,

$$= [w_1(a_1,\ldots,a_p);i,j] + [\pm\, a_t;i,j]$$

$$= [w_1(a_1,\ldots,a_p) \pm a_t;i,j], \text{ by } 2.3\ (1),$$

$$= [w(a_1,\ldots,a_p);i,j].$$

The result now follows by induction.

Theorem 3.24: Let R be a near-ring with identity and let I be an ideal of R. Then $(I,+) \in V$ if and only if $(I^*,+) \in V$.

Proof: Only the sufficient condition needs proof, as the necessary condition has been proved in theorem 3.22.

Let $a_1,\ldots,a_p \in I$. Following the setting of theorem 3.22, we must show that $w(a_1,\ldots,a_p) = 0$. Since $[a_1;i,j],\ldots,[a_p;i,j] \in I^+$, by definition of I^+, and $I^+ \subseteq I^*$, by result 2.10, so $[a_1;i,j],\ldots,[a_p;i,j] \in I^*$, and $w([a_1;i,j],\ldots,[a_p;i,j]) = \mathbf{0}$, as $(I^*,+) \in V$. Therefore $[w(a_1,\ldots,a_p);i,j] = \mathbf{0}$, by lemma 3.23.

Now since R has an identity element and $\mathbf{0}$ can be written as $[0;i,j]$, so by lemma 3.1, we get $w(a_1,\ldots,a_p) = 0$. Therefore the law $w(v_1,\ldots,v_p)$ holds in $(I,+)$. This is true for all the laws of V. Thus $(I,+) \in V$.

Theorem 3.25: Let R be a near-ring and let I be an ideal of R. If $(I,+) \in V$, then $(I^+,+) \in V$.

Proof: Let $\mathcal{A}_1,\ldots,\mathcal{A}_p \in I^+$. Since $\mathcal{A}_1,\ldots\mathcal{A}_p \in I^*$, as $I^+ \subseteq I^*$ by result 2.10, and $w(\mathcal{A}_1,\ldots,\mathcal{A}_p) = \mathbf{0}$, by theorem 3.22, therefore the law $w(v_1,\ldots,v_p)$ holds in $(I^+,+)$. This is true for all the laws of V. Thus $(I^+,+) \in V$.

Theorem 3.26: Let R be a near-ring with identity. Then $(I,+) \in V$ if and only if $(I^+,+) \in V$.

Proof: Only the converse needs proof but we omit it as it is exactly the same way as that of theorem 3.24.

The following results are some immediate consequences of theorem 3.22, theorem 3.25, theorem 3.24 and theorem 3.26.

Corollary 3.27: Let R be a near-ring. Then

(1) If $(I,+)$ is abelian then $(I^+,+)$ and $(I^*,+)$ are abelian.

(2) If $(I,+)$ is soluble then $(I^+,+)$ and $(I^*,+)$ are soluble.

(3) If $(I,+)$ is nilpotent then $(I^+,+)$ and $(I^*,+)$ are nilpotent.

(4) If $(I,+)$ is $\bar{c}$-nilpotent then $(I^+,+)$ and $(I^*,+)$ are $\bar{c}$-nilpotent.

Corollary 3.28: Let R be a near-ring with identity. Then

(1) $(I,+)$ is abelian if and only if $(I^+,+)$ is abelian.

(2) $(I,+)$ is abelian if and only if $(I^*,+)$ is abelian.

(3) $(I,+)$ is soluble if and only if $(I^+,+)$ is soluble.

(4) $(I,+)$ is soluble if and only if $(I^*,+)$ is soluble.

(5) $(I,+)$ is nilpotent if and only if $(I^+,+)$ is nilpotent.

(6) $(I,+)$ is nilpotent if and only if $(I^*,+)$ is nilpotent.

(7) $(I,+)$ is $\bar{c}$-nilpotent if and only if $(I^+,+)$ is $\bar{c}$-nilpotent.

(8) $(I,+)$ is $\bar{c}$-nilpotent if and only if $(I^*,+)$ is $\bar{c}$-nilpotent.

The next result we wish to prove is:

Theorem 3.29: If R is distributive over I then $\mathbb{M}_n(R)$ is distributive over I^*.

The proof will be based on the following lemmas.

Lemma 3.30: If R is distributive over I, then

$$xa + yb = yb + xa$$

where $x, y \in R$ and $a, b \in I$.

Proof: $$(x + y)(b + a) = x(b + a) + y(b + a)$$

by right distributivity of R,

$$= xb + xa + yb + ya$$

as R is distributive over I.

Now by using the hypothesis first, we get

$$(x + y)(b + a) = (x + y)b + (x + y)a$$

$$= xb + yb + xa + ya$$

by right distributivity of R.

Hence $xa + yb = yb + xa$.

Lemma 3.31: If R is distributive over I then

$$[x;i,j]\alpha_1 + [y;k,l]\alpha_2 = [y;k,l]\alpha_2 + [x;i,j]\alpha_1$$

where $x, y \in R$, $\alpha_1, \alpha_2 \in I^n$ and $1 \leqslant i,j,k,l \leqslant n$.

Proof: Let $\alpha_1 = \langle a_1,\ldots,a_n\rangle$ and $\alpha_2 = \langle b_1,\ldots,b_n\rangle$ where $a_1,\ldots,a_n,b_1,\ldots,b_n \in I$, then

$$[x;i,j]\alpha_1 + [y;i,l]\alpha_2 = \langle 0,\ldots,xa_j + yb_l,\ldots,0\rangle$$

with $xa_j + yb_l$ in the i-th place,

$$= \langle 0,\ldots,yb_l + xa_j,\ldots,0\rangle$$

with $yb_l + xa_j$ in the i-th place, by lemma 3.30,

$$= [y;i,l]\alpha_2 + [x;i,j]\alpha_1.$$

The case when i differs from k is obvious.

Lemma 3.32: Let R be a zero-symmetric near-ring. If R is distributive over I, an ideal of R, then

$$\mathcal{X}\alpha_1 + \mathcal{Y}\alpha_2 = \mathcal{Y}\alpha_2 + \mathcal{X}\alpha_1$$

where $\mathcal{X}, \mathcal{Y} \in \mathbb{M}_n(R)$ and $\alpha_1, \alpha_2 \in I^n$.

Proof: We use induction on $w(\mathcal{X}) + w(\mathcal{Y})$. If $w(\mathcal{X}) + w(\mathcal{Y}) = 2$, then $w(\mathcal{X}) = w(\mathcal{Y}) = 1$ and the result follows from lemma 3.31.

Suppose $\mathcal{X}\alpha_1 + \mathcal{Y}\alpha_2 = \mathcal{Y}\alpha_2 + \mathcal{X}\alpha_1$, whenever $w(\mathcal{X}) + w(\mathcal{Y}) < m$, $m \in \mathbb{N}$, $m \geq 3$. If $w(\mathcal{X}) + w(\mathcal{Y}) = m$, then we have the following possibilities:

$\mathcal{X} = \mathcal{X}_1 + \mathcal{X}_2$ or $\mathcal{X} = \mathcal{X}_1\mathcal{X}_2$ or $\mathcal{Y} = \mathcal{Y}_1 + \mathcal{Y}_2$ or $\mathcal{Y} = \mathcal{Y}_1\mathcal{Y}_2$

where $w(\mathcal{X}_1)$, $w(\mathcal{X}_2)$, $w(\mathcal{Y}_1)$, $w(\mathcal{Y}_2) < m$.

It sufficies to discuss only first two possibilities. The other two are similar.

(1) Let $\mathcal{X} = \mathcal{X}_1 + \mathcal{X}_2$. Then

$$\begin{aligned}\mathcal{X}\alpha_1 + \mathcal{Y}\alpha_2 &= (\mathcal{X}_1 + \mathcal{X}_2)\alpha_1 + \mathcal{Y}\alpha_2 \\ &= \mathcal{X}_1\alpha_1 + \mathcal{X}_2\alpha_1 + \mathcal{Y}\alpha_2 \\ &= \mathcal{X}_1\alpha_1 + \mathcal{Y}\alpha_2 + \mathcal{X}_2\alpha_1\end{aligned}$$

by the induction hypothesis, as $w(\mathcal{Y}) + w(\mathcal{X}_2) < m$,

$$= \mathcal{Y}\alpha_2 + \mathcal{X}_1\alpha_1 + \mathcal{X}_2\alpha_1$$

by the induction hypothesis, as $w(\mathcal{Y}) + w(\mathcal{X}_1) < m$,

$$= \mathcal{Y}\alpha_2 + (\mathcal{X}_1 + \mathcal{X}_2)\alpha_1$$

$$= \mathcal{Y}\alpha_2 + \mathcal{X}\alpha_1$$

(2) Let $\mathcal{X} = \mathcal{X}_1\mathcal{X}_2$. Then

$$\mathcal{X}\alpha_1 + \mathcal{Y}\alpha_2 = (\mathcal{X}_1\mathcal{X}_2)\alpha_1 + \mathcal{Y}\alpha_2$$

$$= \mathcal{X}_1(\mathcal{X}_2\alpha_1) + \mathcal{Y}\alpha_2$$

$$= \mathcal{Y}\alpha_2 + \mathcal{X}_1(\mathcal{X}_2\alpha_1)$$

by the induction hypothesis, as $w(\mathcal{X}_1) + w(\mathcal{Y}) < m$ and $\mathcal{X}_2\alpha_1 \in I^n$,

(see results 2.4, 2.6 and 1.31),

$$= \mathcal{Y}\alpha_2 + (\mathcal{X}_1\mathcal{X}_2)\alpha_1$$

$$= \mathcal{Y}\alpha_2 + \mathcal{X}\alpha_1.$$

The result now follows by induction.

<u>Lemma</u> <u>3.33</u>: If R is distributive over I, then

$$[x;i,j]\ (\alpha_1 + \alpha_2) = [x;i,j]\alpha_1 + [x;i,j]\alpha_2$$

where $x \in R$, α_1, $\alpha_2 \in I^n$ and $1 \leqslant i,j \leqslant n$.

Proof: $[x;i,j]\ (\alpha_1 + \alpha_2) = [x;i,j]\ \langle a_1 + b_1, \dots, a_n + b_n\rangle$

$$= \langle 0, .., x(a_j + b_j), .., 0\rangle$$

with $x(a_j + b_j)$ in the i-th place,

$$= \langle 0, .., xa_j + xb_j, .., 0\rangle$$

with $xa_j + xb_j$ in the i-th place,

$$= [x;i,j]\alpha_1 + [x;i,j]\alpha_2.$$

Lemma 3.34: Let R be a zero-symmetric near-ring. If R is distributive over I, then $\mathbb{M}_n(R)$ is distributive over I^n.

Proof: Let $\mathcal{X} \in \mathbb{M}_n(R)$ and $\alpha_1, \alpha_2 \in I^n$. We must show that $\mathcal{X}(\alpha_1 + \alpha_2) = \mathcal{X}\alpha_1 + \mathcal{X}\alpha_2$. We use induction on the weight, $w(\mathcal{X})$, of $\mathcal{X}$. If $w(\mathcal{X}) = 1$, then the result follows from lemma 3.33. Assume that the lemma holds for all matrices of $\mathbb{M}_n(R)$ having weight less than m, $m \in \mathbb{N}$, $m \geqslant 2$. If $w(\mathcal{X}) = m$, then there are two possibilities: $\mathcal{X} = \mathcal{X}_1 + \mathcal{X}_2$ or $\mathcal{X} = \mathcal{X}_1\mathcal{X}_2$ where $w(\mathcal{X}_1)$, $w(\mathcal{X}_2) < m$.

(1) Let $\mathcal{X} = \mathcal{X}_1 + \mathcal{X}_2$.

$$\mathcal{X}(\alpha_1 + \alpha_2) = (\mathcal{X}_1 + \mathcal{X}_2)\ (\alpha_1 + \alpha_2)$$

$$= \mathcal{X}_1(\alpha_1 + \alpha_2) + \mathcal{X}_2(\alpha_1 + \alpha_2)$$

$$= \mathcal{X}_1\alpha_1 + \mathcal{X}_1\alpha_2 + \mathcal{X}_2\alpha_1 + \mathcal{X}_2\alpha_2$$

by the induction hypothesis,

$$= \mathcal{X}_1\alpha_1 + \mathcal{X}_2\alpha_1 + \mathcal{X}_1\alpha_2 + \mathcal{X}_2\alpha_2$$

by lemma 3.32,

$$= (\mathcal{X}_1 + \mathcal{X}_2)\alpha_1 + (\mathcal{X}_1 + \mathcal{X}_2)\alpha_2$$

$$= \mathcal{X}\alpha_1 + \mathcal{X}\alpha_2.$$

(2) Let $\mathcal{X} = \mathcal{X}_1\mathcal{X}_2$.

$$\mathcal{X}(\alpha_1 + \alpha_2) = (\mathcal{X}_1\mathcal{X}_2)(\alpha_1 + \alpha_2)$$

$$= \mathcal{X}_1(\mathcal{X}_2(\alpha_1 + \alpha_2))$$

$$= \mathcal{X}_1(\mathcal{X}_2\alpha_1 + \mathcal{X}_2\alpha_2)$$

by the induction hypothesis,

$$= \mathcal{X}_1(\mathcal{X}_2\alpha_1) + \mathcal{X}_1(\mathcal{X}_2\alpha_2)$$

by the induction hypothesis, as $\mathcal{X}_2\alpha_1$, $\mathcal{X}_2\alpha_2 \in I^n$, because I^n is an $\mathbb{M}_n(R)$-submodule of R^n (see results 2.6 and 1.31),

$$= (\mathcal{X}_1\mathcal{X}_2)\alpha_1 + (\mathcal{X}_1\mathcal{X}_2)\alpha_2$$

$$= \mathcal{X}\alpha_1 + \mathcal{X}\alpha_2.$$

The result now follows by induction.

Proof of Theorem 3.29: Let $\mathcal{X} \in \mathbb{M}_n(R)$, $\mathcal{A}$, $\mathcal{B} \in I^*$ and $\alpha \in R^n$. Then

$$\mathcal{X}(\mathcal{A} + \mathcal{B})\alpha = \mathcal{X}(\mathcal{A}\alpha + \mathcal{B}\alpha)$$

$$= \mathcal{X}(\mathcal{A}\alpha) + \mathcal{X}(\mathcal{B}\alpha)$$

by lemma 3.34, as $\mathcal{A}\alpha$, $\mathcal{B}\alpha \in I^n$, by definition of I^*,

$$= (\mathcal{X}\mathcal{A})\alpha + (\mathcal{X}\mathcal{B})\alpha$$

$$= (\mathcal{X}\mathcal{A} + \mathcal{X}\mathcal{B})\alpha.$$

Hence $\mathcal{X}(\mathcal{A} + \mathcal{B}) = \mathcal{X}\mathcal{A} + \mathcal{X}\mathcal{B}$. This completes the proof.

<u>Corollary 3.35</u>: Let R be a zero-symmetric near-ring. If R is distributive over I, then $\mathbb{M}_n(R)$ is distributive over I^+.

Proof: Let $\mathcal{X} \in \mathbb{M}_n(R)$ and $\mathcal{A}$, $\mathcal{B} \in I^+$. We need to show that $\mathcal{X}(\mathcal{A} + \mathcal{B}) = \mathcal{X}\mathcal{A} + \mathcal{X}\mathcal{B}$. Since $I^+ \subseteq I^*$ (see result 2.10), and $\mathbb{M}_n(R)$ is distributive over I^*(see theorem 3.29), therefore we get the result that we want.

<u>Corollary 3.36</u>: Let R be a zero-symmetric near-ring with identity. If $\mathcal{I}$ is an ideal of $\mathbb{M}_n(R)$ and R is distributive over $\mathcal{I}_*$ then $\mathbb{M}_n(R)$ is distributive over $\mathcal{I}$.

Proof: Let $\mathcal{X} \in \mathbb{M}_n(R)$ and $\mathcal{A}$, $\mathcal{B} \in \mathcal{I}$. Since $\mathcal{I} \subseteq (\mathcal{I}_*)^*$ (see result 2.23) so $\mathcal{A}$, $\mathcal{B} \in (\mathcal{I}_*)^*$. Now as $\mathbb{M}_n(R)$ is distributive over $(\mathcal{I}_*)^*$, by theorem 3.29, we get $\mathcal{X}(\mathcal{A} + \mathcal{B}) = \mathcal{X}\mathcal{A} + \mathcal{X}\mathcal{B}$. This leads to the desired result.

We now apply this corollary to establish the following theorem.

Theorem 3.37: Let R be a zero-symmetric near-ring with identity and let $\mathcal{I}$ be an ideal of $\mathbb{M}_n(R)$. Then R is distributive over $\mathcal{I}_*$ if and only if $\mathbb{M}_n(R)$ is distributive over $\mathcal{I}$.

Proof: There remains only to prove the sufficient condition as the necessary condition follows from corollary 3.36.

Let $x \in R$ and $a, b \in \mathcal{I}_*$. We have $[a;1,1]$, $[b;1,1] \in \mathcal{I}$

(see result 2.20). Now

$[x;1,1]([a;1,1] + [b;1,1]) = [x;1,1][a;1,1] + [x;1,1][b;1,1]$, as $\mathbb{M}_n(R)$ is distributive over $\mathcal{I}$. So

$[x;1,1][a + b;1,1] = [xa;1,1] + [xb;1,1]$, by results 2.3(3) and 2.3(1), and $[x(a + b);1,1] = [xa + xb;1,1]$, again by results 2.3(3) and 2.3(1). Hence $x(a + b) = xa + xb$, by lemma 3.1.

Exactly the same method of proof enables us to show the converse of theorem 3.29, and that of corollary 3.35. To avoid repetition we omit them.

Theorem 3.38: Let R be a zero-symmetric near-ring with identity and let I be an ideal of R. Then R is distributive over I if and only if $\mathbb{M}_n(R)$ is distributive over I^+.

Theorem 3.39: Let R be a zero-symmetric near-ring with identity, I an ideal of R. Then R is distributive over I if and only if $\mathbb{M}_n(R)$ is distributive over I^*.

It will be recalled from definition 2.1 that $\mathbb{M}_n(R)$ is a near-ring generated by the set $\{[r;i,j] : r \in R, 1 \leqslant i,j \leqslant n\}$. If I is an ideal of R, then $\overline{\mathbb{M}}_n(I)$ is a subnear-ring of $\mathbb{M}_n(R)$ generated by the set $\{[a;i,j] : a \in I, 1 \leqslant i,j \leqslant n\}$.

Since $I^+ \subseteq I^*$, the following lemma is obvious.

<u>Lemma 3.40</u>: Let R be a near-ring, and I be an ideal of R. Then $\overline{\mathbb{M}}_n(I) \subseteq I^+ \subseteq I^*$.

We can now easily prove the following theorem:

<u>Theorem 3.41</u>: Let R be a zero-symmetric near-ring with identity. Then R is distributive over I if and only if $\mathbb{M}_n(R)$ is distributive over $\overline{\mathbb{M}}_n(I)$.

Proof: Let $\mathcal{X} \in \mathbb{M}_n(R)$ and $\mathcal{A}$, $\mathcal{B} \in \overline{\mathbb{M}}_n(I)$. Since $\mathcal{A}$, $\mathcal{B} \in I^*$, by lemma 3.40, and $\mathcal{X}(\mathcal{A} + \mathcal{B}) = \mathcal{X}\mathcal{A} + \mathcal{X}\mathcal{B}$ by theorem 3.29, therefore $\mathbb{M}_n(R)$ is distributive over $\overline{\mathbb{M}}_n(I)$.

For the converse, let $x \in R$ and $a, b \in I$. Then by definition of $\overline{\mathbb{M}}_n(I)$, $[a;i,j]$, $[b;i,j] \in \overline{\mathbb{M}}_n(I)$. The rest of the proof is exactly the same as that of theorem 3.37 and is therefore omitted.

We conclude this chapter by answering the question posed in Meldrum and Van Der Walt [6]: Does, in general, $\mathbb{M}_n(R)$ possess ideals which are not full? The answer is "Yes".

We first establish the following lemmas:

Lemma 3.42: Let R be a zero-symmetric near-ring with identity. If $I^+ = I^*$, for every ideal I of R, then all ideals of $\mathbb{M}_n(R)$ are full.

Proof: Let $\mathcal{I}$ be an ideal of $\mathbb{M}_n(R)$. We wish to show that $\mathcal{I} = (\mathcal{I}_*)^*$ (see definition 2.8). Since $\mathcal{I}_*$ is an ideal of R (see result 2.22), so by hypothesis, $(\mathcal{I}_*)^* = (\mathcal{I}_*)^+$ and since $(\mathcal{I}_*)^+ \subseteq \mathcal{I} \subseteq (\mathcal{I}_*)^*$ (see result 2.25), therefore $(\mathcal{I}_*)^* = \mathcal{I}$.

Lemma 3.43: Let R be a zero-symmetric near-ring with identity. Then $I^+ = I^*$, for each ideal I of R, if and only if all ideals of $\mathbb{M}_n(R)$ are full.

Proof: The necessary condition follows from lemma 3.42. For the converse, let us take an ideal I^+ of $\mathbb{M}_n(R)$. Since I^+ is a full ideal, so $I^+ = L^*$, for some ideal L of R, by definition of a full ideal. We aim to show that I = L.

Let $a \in I$ then $[a;1,1] \in I^+$, by definition of I^+.

Since $I^+ = L^*$, so $[a;1,1] \in L^*$. Therefore $[a;1,1]\varepsilon_1 \in L^n$, where $\varepsilon_1 = <1,0,\ldots,0>$. This forces $a \in L$ and so $I \subseteq L$.

It is known that $L^+ \subseteq L^*$ (see result 2.10). So $L^+ \subseteq I^+$, as $I^+ = L^*$.

Now since $(\ )^+$ is an order preserved injection (see result 2.18), we get $L \subseteq I$.

This completes the proof.

Corollary 3.44: Let R be a zero-symmetric near-ring with identity. If there exists an ideal I of R such that $I^+ \neq I^*$, then not all ideals of $\mathbb{M}_n(R)$ are full.

Proof: Suppose that all ideals of $\mathbb{M}_n(R)$ are full. Then $I^+ = I^*$, for each ideal I of R, by lemma 3.43. This contradicts the hypothesis.

The existence of a near-ring R and an ideal I of R such that $I^+ \subsetneqq I^*$ was shown in Van Der Walt [14] (see result 2.26). This enables us to answer the above mentioned question .

<u>Theorem</u> <u>3.45</u>: In general, $\mathbb{M}_n(R)$ possesses ideals which are not full.

Proof: The result follows from corollary 3.44 and the remark made earlier.

CHAPTER 4

DISTRIBUTIVELY-GENERATED MATRIX NEAR-RINGS

Before proceeding to the discussion on d.g. matrix near-rings, we remark that, for the sake of brevity, we use the same notation R for a d.g. near-ring in place of (R,S), if S is not important.

One may ask what can be said about the elements of $\mathbb{M}_n(R)$, when R is a d.g. near-ring. The answer is:

Theorem 4.1: If R is a d.g. near-ring, then every element of $\mathbb{M}_n(R)$ is simply the sum of elementary matrices of $\mathbb{M}_n(R)$.

Proof: Suppose R is distributively-generated by $(S,.) \subseteq (R_d,.)$.

Recall that if s is a distributive element of R, then $[s;i,j]$ is distributive in $\mathbb{M}_n(R)$ (see result 2.3 (8)).

It suffices to show that $[x;i,j]([y;k,l] + [z;k',l'])$ can be written as a sum of elementary matrices, where x, y, z, are in R and $1 \leqslant i,j,k,l,k',l' \leqslant n$.

Since $x = \varepsilon_1 s_1 + \ldots + \varepsilon_m s_m$ where $\varepsilon_t = \pm 1$, $s_t \in S$ and $1 \leqslant t \leqslant m$ so

$$[x;i,j]([y;k,l] + [z;k',l'])$$

$$= [\varepsilon_1 s_1 + \ldots + \varepsilon_m s_m;i,j]([y;k,l] + [z;k',l'])$$

$$= (\varepsilon_1[s_1;i,j] + \ldots + \varepsilon_m[s_m;i,j])([y;k,l] + [z;k',l'])$$

by result 2.3 (1) and result 2.3 (4),

$$= \varepsilon_1[s_1;i,j]([y;k,l] + [z;k',l']) + \ldots\ldots +$$

$$\varepsilon_m[s_m;i,j]([y;k,l] + [z;k',l'])$$

by the right distributive law,

$$= \varepsilon_1([s_1;i,j][y;k,l] + [s_1;i,j][z;k';l']) +\ldots+$$

$$+ \varepsilon_m([s_m;i,j][y;k,l]) + [s_m;i,j][z;k',l'])$$

by the remark made earlier.

Now result 2.3 (3) supplies the rest of what we need.

There are some remarks to be made:

Remarks: (1) Let R be a d.g. near-ring. If $\mathcal{X} \in \mathbb{M}_n(R)$, and $w(\mathcal{X}) = m$, $m \in \mathbb{N}$, $m \geqslant 2$, then $\mathcal{X} = \mathcal{X}_1 + \mathcal{X}_2$ where $w(\mathcal{X}_1)$, $w(\mathcal{X}_2) < w(\mathcal{X})$.

(2) It is known that if R is a d.g. near-ring, then so is $\mathbb{M}_n(R)$ (see result 2.5). This result was proved in Meyer [8], but there is no need to discuss the second possibility :

$\mathcal{U} = \mathcal{X}\mathcal{Y}$ with $w(\mathcal{X})$, $w(\mathcal{Y}) < w(\mathcal{U})$ (see corollary 1.17, Meyer [8]).

We now combine result 1.89 and corollary 3.12 to get the following result.

Corollary 4.2: If R is a d.g. near-ring such that (R,+) is $\underline{c}$-nilpotent, then $(\mathbb{M}_n(R),+)$ is $\underline{c}$-nilpotent.

Proof: If (R,+) is $\underline{c}$-nilpotent,

then (R,+) is $\bar{c}$-nilpotent, by result 1.89,

so $(\mathbb{M}_n(R),+)$ is $\bar{c}$-nilpotent, by corollary 3.12,

therefore $(\mathbb{M}_n(R),+)$ is $\underline{c}$-nilpotent, by result 1.89 and result 2.5.

Corollary 4.3: Let R be a d.g. near-ring with identity. Then (R,+) is $\underline{c}$-nilpotent if and only if $(\mathbb{M}_n(R),+)$ is $\underline{c}$-nilpotent.

Proof: Only the sufficient condition needs proof.

If $(\mathbb{M}_n(R),+)$ is $\underline{c}$-nilpotent,

then $(\mathbb{M}_n(R),+)$ is $\bar{c}$-nilpotent, by result 1.89 and result 2.5,

so $(R,+)$ is $\bar{c}$-nilpotent, by corollary 3.13,

therefore $(R,+)$ is $\underline{c}$-nilpotent, by result 1.89.

The following result is important.

Theorem 4.4: If R is distributive, then $(\mathbb{M}_n(R),+)$ is abelian.

Proof: We have

$[x;i,j] + [y;k,l] = [y;k,l] + [x;i,j]$ if $i \neq k$ (see result 2.3 (2)).

So we only need to show that this is also true when $i = k$.

Let $\alpha = < r_1,\dots,r_n > \in R^n$. Then

$([x;i,j] + [y;i,l])\alpha = [x;i,j]\alpha + [y;i,l]\alpha$

$= < 0,..,xr_j,..,0 > + <0,..,yr_l,..,0 >$

with xr_j and yr_l in the i-th places,

$= < 0,..,xr_j + yr_l,..,0 >$

with $xr_j + yr_l$ in the i-th position of the vector,

$= < 0,..,yr_l + xr_j,..,0 >$

by result 1.59,

$= ([y;i,l] + [x;i,j])\alpha.$

Hence $[x;i,j] + [y;i,l] = [y;i,l] + [x;i,j]$.

The next series of results is an immediate consequence of the series of lemmas 3.31 to 3.34. They can be proved independently by following exactly a similar method to that of the proofs of lemmas

3.31 to 3.34, respectively.

Lemma 4.5: If R is distributive, then

$$[x;i,j]\alpha + [y;k,l]\beta = [y;k,l]\beta + [x;i,j]\alpha$$

where x, y $\in$ R, $1 \leqslant i,j,k,l \leqslant n$ and α, $\beta \in R^n$.

Lemma 4.6: If R is distributive, then

$$\mathcal{X}\alpha + \mathcal{Y}\beta = \mathcal{Y}\beta + \mathcal{X}\alpha$$

where $\mathcal{X}$, $\mathcal{Y} \in \mathbb{M}_n(R)$ and α, $\beta \in R^n$.

Lemma 4.7: If R is distributive, then

$$[x;i,j](\alpha + \beta) = [x;i,j]\alpha + [x;i,j]\beta$$

where $x \in R$, α, $\beta \in R^n$ and $1 \leqslant i,j \leqslant n$.

Lemma 4.8: If R is distributive, then $\mathbb{M}_n(R)$ is distributive over R^n.

Theorem 4.9: If R is distributive, then $\mathbb{M}_n(R)$ is distributive.

Proof: Let $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z} \in \mathbb{M}_n(R)$ and $\alpha \in R^n$. Then

$$\begin{aligned}
\mathcal{X}(\mathcal{Y} + \mathcal{Z})\alpha &= \mathcal{X}(\mathcal{Y}\alpha + \mathcal{Z}\alpha)\\
&= \mathcal{X}(\sigma_1 + \sigma_2), \text{ where } \sigma_1 = \mathcal{Y}\alpha,\ \sigma_2 = \mathcal{Z}\alpha,\\
&= \mathcal{X}\sigma_1 + \mathcal{X}\sigma_2, \text{ by lemma 4.8},\\
&= \mathcal{X}(\mathcal{Y}\alpha) + \mathcal{X}(\mathcal{Z}\alpha)\\
&= (\mathcal{X}\mathcal{Y} + \mathcal{X}\mathcal{Z})\alpha.
\end{aligned}$$

Hence $\mathcal{X}(\mathcal{Y} + \mathcal{Z}) = \mathcal{X}\mathcal{Y} + \mathcal{X}\mathcal{Z}$.

Note: This result follows also from theorem 3.29 as $R^* = \mathbb{M}_n(R)$.

Evidently, the preceding result and theorem 4.4 have the following consequences.

Thoerem 4.10: If R is distributive, then $\mathbb{M}_n(R)$ is a ring.

Corollary 4.11: If R is a ring, then $\mathbb{M}_n(R)$ is a ring.

Remark: The sufficiency of this result is not true, in general, as there exists a near-ring R which is not a ring but the matrix near-ring $\mathbb{M}_n(R)$ is a ring (see example 3.7 and theorem 4.10).

Theorem 4.12: If R has an identity element, then R is a ring if and only if $\mathbb{M}_n(R)$ is a ring.

Proof: The proof of the converse is very easy, and is therefore omitted.

The following result will be useful in the sequel.

Lemma 4.13: Let R be a d.g. near-ring with identity. Then

$$\mathcal{X}(\alpha + \beta) = \mathcal{X}\alpha + \mathcal{X}\beta + \sigma$$

where $\mathcal{X} \in \mathbb{M}_n(R)$, $\alpha, \beta \in R^n$ and $\sigma \in (D^1(R))^n$.

Proof: We use induction on the weight, $w(\mathcal{X})$, of $\mathcal{X}$.

Let $\alpha = \langle x_1,\ldots,x_n\rangle$ and $\beta = \langle y_1,\ldots,y_n\rangle$. Then

$$[x;i,j](\alpha + \beta) = [x;i,j]\,\langle x_1 + y_1,\ldots, x_n + y_n\rangle$$

$$= \langle 0,\ldots,x(x_j + y_j),\ldots,0\rangle$$

with $x(x_j + y_j)$ in the i-th place,

$$= \langle 0,\ldots, xx_j + xy_j + a,\ldots,0\rangle$$

where $a \in D^1(R)$,

$$= [x;i,j]\alpha + [x;i,j]\beta + \gamma$$

where $\gamma = \langle 0,\ldots,a,\ldots,0\rangle \in (D^1(R))^n$.

Next we assume that $\mathcal{Y}(\alpha + \beta) = \mathcal{Y}\alpha + \mathcal{Y}\beta + \zeta$, where $\zeta \in (D^1(R))^n$ and $w(\mathcal{Y}) < m$. If $w(\mathcal{X}) = m$, then $\mathcal{X} = \mathcal{X}_1 + \mathcal{X}_2$ with $w(\mathcal{X}_1)$, $w(\mathcal{X}_2) < m$. Now

$$\mathcal{X}(\alpha + \beta) = (\mathcal{X}_1 + \mathcal{X}_2)(\alpha + \beta)$$

$$= \mathcal{X}_1(\alpha + \beta) + \mathcal{X}_2(\alpha + \beta)$$

$$= \mathcal{X}_1\alpha + \mathcal{X}_1\beta + \zeta_1 + \mathcal{X}_2\alpha + \mathcal{X}_2\beta + \zeta_2$$

where $\zeta_1, \zeta_2 \in (D^1(R))^n$, by the induction hypothesis,

$$= \mathcal{X}_1\alpha + \mathcal{X}_2\alpha + \mathcal{X}_1\beta + \mathcal{X}_2\beta + \zeta_1 + \zeta_2 + \zeta_3$$

where $\zeta_3 \in (\delta_1(R))^n$ (see result 1.5 and definition 1.6),

$$= (\mathcal{X}_1 + \mathcal{X}_2)\alpha + (\mathcal{X}_1 + \mathcal{X}_2)\beta + \zeta_4$$

where $\zeta_4 = \zeta_1 + \zeta_2 + \zeta_3 \in (D^1(R))^n$, as $\delta_1(R) = D^1(R)$, by result 1.69

$$= \mathcal{X}\alpha + \mathcal{X}\beta + \zeta_4.$$

The result now follows by induction.

The rest of this chapter is devoted to a discussion on the relationship between commutator, distributor and lower central series of R and those of $\mathbb{M}_n(R)$.

It is necessary to define, first, the commutator subgroup of $(\mathbb{M}_n(R),+)$.

Let $[x;i,j]$ and $[y;k,l] \in \mathbb{M}_n(R)$ for any x, $y \in R$, and $1 \leqslant i,j,k,l \leqslant n$. The commutator of $[x;i,j]$ and $[y;k,l]$, denoted by $\mathcal{C}$, is defined as:

$\mathcal{C} := ([x;i,j],[y;k,l]) := -[x;i,j] - [y;k,l] + [x;i,j] + [y;k,l]$

(see definition 1.1).

Note that $\mathcal{C} = 0$, if $i \neq k$, by result 2.3 (2).

<u>Definition 4.14</u>: Let R be a d.g. near-ring. The derived group (or commutator subgroup) of $(\mathbb{M}_n(R),+)$, denoted by $\delta_1(\mathbb{M}_n(R))$ is defined as:

$$\delta_1(\mathbb{M}_n(R)) := \mathrm{Gp}\langle([x;i,j],[y;k,l]) : x,\ y \in R,\ 1 \leqslant i,j,k,l \leqslant n\rangle^{\mathbb{M}_n(R)}$$

(see definition 1.2).

<u>Lemma 4.15</u>: Let R be a d.g. near-ring. Then

$[(x,y);i,j] = ([x;i,j],[y;i,j])$

where $x,\ y \in R$ and $1 \leqslant i,j \leqslant n$.

Proof: $[(x,y);i,j] = [-x-y+x+y;i,j]$

$= -[x;i,j] - [y;i,j] + [x;i,j] + [y;i,j]$

by result 2.3 (1) and result 2.3 (4),

$= ([x;i,j],[y;i,j])$.

<u>Lemma 4.16</u>: Let I_1 and I_2 be ideals of a d.g. near-ring R. Then $\bar{\mathbb{M}}_n((I_1,I_2)) \subseteq (I_1^+,I_2^+)$.

Proof: Recall that (I_1,I_2) is an ideal of (R,S) (see result 1.61), and $\bar{\mathbb{M}}_n((I_1,I_2))$ is a subnear-ring of $\mathbb{M}_n(R)$ generated by the set $\{[(a,b);i,j] : a \in I_1,\ b \in I_2 \text{ and } 1 \leqslant i,j \leqslant n\}$.

Let $\mathcal{A} \in \bar{\mathbb{M}}_n((I_1,I_2))$. We use induction on the weight, $w(\mathcal{A})$, of $\mathcal{A}$.

If $w(\mathcal{A}) = 1$ and $\mathcal{A} = [x;i,j]$, where $x \in (I_1,I_2)$ and $1 \leqslant i,j \leqslant n$, then $\mathcal{A} = [\varepsilon_1(a_1,b_1) + \dots + \varepsilon_m(a_m,b_m);i,j]$, where $a_t \in I_1$, $b_t \in I_2$ for $1 \leq t \leq m$. By result 2.3 (1) and (4), we get

$\mathcal{A} = \varepsilon_1\ [(a_1,b_1);i,j] + \dots + \varepsilon_m\ [(a_m,b_m);i,j]$

$= \varepsilon_1\ ([a_1;i,j],(b_1;i,j]) + \dots + \varepsilon_m\ ([a_m;i,j],[b_m;i,j])$

by lemma 4.15. Therefore $\mathscr{A} \in (I_1^+, I_2^+)$, as $[a_t;i,j] \in I_1^+$ and $[b_t;i,j] \in I_2^+$ for $1 \leq t \leq m$.

Now suppose that the result has been proved for all elements of $\bar{\mathbb{M}}_n((I_1,I_2))$ of weight less than m, $m \in \mathbb{N}$, $m \geqslant 2$. If $w(\mathscr{A}) = m$, then $\mathscr{A} = \mathscr{A}_1 + \mathscr{A}_2$ or $\mathscr{A} = \mathscr{A}_1\mathscr{A}_2$, with $w(\mathscr{A}_1)$, $w(\mathscr{A}_2) < m$.

(1) Let $\mathscr{A} = \mathscr{A}_1 + \mathscr{A}_2$. Then obviously $\mathscr{A} \in (I_1^+, I_2^+)$, as $\mathscr{A}_1$, $\mathscr{A}_2 \in (I_1^+, I_2^+)$, by the induction hypothesis.

(2) Let $\mathscr{A} = \mathscr{A}_1\mathscr{A}_2$. Then since $\mathscr{A}_1 \in (I_1^+, I_2^+)$, by the induction hypothesis, and $(I_1^+, I_2^+)\ \mathbb{M}_n(R) \subseteq (I_1^+, I_2^+)$, by result 1.31 because $\mathbb{M}_n(R)$ is a zero-symmetric near-ring (see results 1.57 and 2.5), and (I_1^+, I_2^+) is an ideal of $\mathbb{M}_n(R)$ (see result 1.61), therefore $\mathscr{A} \in (I_1^+, I_2^+)$. The result now follows by induction.

Before proceeding to the next lemma, we remark that I^+ can be simplified in the d.g. case as follows:

$$I^+ = \mathrm{Id}\ \langle \bar{\mathbb{M}}_n(I) \rangle$$
$$= \mathrm{Gp}\ \langle \mathscr{S}\ \bar{\mathbb{M}}_n(I)\ \mathbb{M}_n(R) \rangle^{\mathbb{M}_n(R)}$$

where $\mathscr{S} := \{[s;i,j] : s \in S,\ 1 \leqslant i,j \leqslant n\}$,

$\mathscr{S}\bar{\mathbb{M}}_n(I)\mathbb{M}_n(R) = \{\mathscr{A}\mathscr{B}\mathscr{C},\ \mathscr{A}\mathscr{B},\ \mathscr{B}\mathscr{C},\ \mathscr{B} : \mathscr{A} \in \mathscr{S},\ \mathscr{B} \in \bar{\mathbb{M}}_n(I),\ \mathscr{C} \in \mathbb{M}_n(R)\}$

(see result 1.73) and $\mathrm{Gp}\ \langle X \rangle^R$ is the normal subgroup of R generated by X.

<u>Lemma 4.17</u>: Let I_1 and I_2 be ideals of a d.g. near-ring R. Then

$$(I_1^+, I_2^+) \supseteq (I_1, I_2)^+.$$

Proof: Since $(I_1^+, I_2^+) \supseteq \mathscr{S}(I_1^+, I_2^+)\mathbb{M}_n(R)$, as (I_1^+, I_2^+) is an ideal of $\mathbb{M}_n(R)$, and $\mathscr{S}(I_1^+, I_2^+)\mathbb{M}_n(R) \supseteq \mathscr{S}\bar{\mathbb{M}}_n((I_1, I_2))\mathbb{M}_n(R)$, as $(I_1^+, I_2^+) \supseteq \bar{\mathbb{M}}_n((I_1, I_2))$, by lemma 4.16, therefore $(I_1^+, I_2^+) \supseteq \mathscr{S}\bar{\mathbb{M}}_n((I_1, I_2))\mathbb{M}_n(R)$. Now since $((I_1^+, I_2^+), +)$ is a normal subgroup of $(\mathbb{M}_n(R), +)$, it follows that

$(I_1^+, I_2^+) \supseteq \mathscr{S}\bar{\mathbb{M}}_n((I_1, I_2))\mathbb{M}_n(R)$ if and only if

$$(I_1^+, I_2^+) \supseteq \mathrm{Gp}\langle \mathscr{S}\bar{\mathbb{M}}_n((I_1, I_2))\mathbb{M}_n(R)\rangle^{\mathbb{M}_n(R)}$$
$$= (I_1, I_2)^+ \text{ (see the remark made earlier).}$$

This result has some useful corollaries.

<u>Corollary</u> <u>4.18</u>: Let R be a d.g. near-ring.

Then $(\delta_1(R))^+ \subseteq \delta_1(\mathbb{M}_n(R))$.

Proof:
$$\begin{aligned}\delta_1(\mathbb{M}_n(R)) &= (\mathbb{M}_n(R), \mathbb{M}_n(R)) \\ &= (R^+, R^+), \text{ as } R^+ = \mathbb{M}_n(R), \\ &\supseteq (R, R)^+, \text{ by lemma 4.17,} \\ &= (\delta_1(R))^+.\end{aligned}$$

<u>Corollary</u> <u>4.19</u>: Let R be a d.g. near-ring, and I be an ideal of R.

Then $(\delta_1(I))^+ \subseteq \delta_1(I^+)$.

Proof: This can be proved in a similar manner to the above result.

<u>Corollary</u> <u>4.20</u>: Let R be a d.g. near-ring with identity. Then

$(R; I_1, I_2)^+ \subseteq (\mathbb{M}_n(R); I_1^+, I_2^+)$.

Proof: Since $(R;I_1,I_2) = (I_1,I_2)$, by result 1.68, ~~and $(\ )^+$ is~~ an i~~njection, by result 2.18,~~ so

$$
\begin{aligned}
(R;I_1,I_2)^+ &= (I_1,I_2)^+ \\
&\subseteq (I_1^+,I_2^+), \text{ by lemma 4.17,} \\
&= (\mathbb{M}_n(R);I_1^+,I_2^+), \text{ by result 1.68.}
\end{aligned}
$$

We are now able to show the relationship between the commutator, distributor and lower central series of R and those of $\mathbb{M}_n(R)$.

<u>Theorem</u> <u>4.21</u>: Let R be a d.g. near-ring. Then

$(\delta_k(R))^+ \subseteq \delta_k(\mathbb{M}_n(R))$ for all $k \geqslant 0$.

Proof: We use induction on k. For $k = 0$, the result is obvious as $R^+ = \mathbb{M}_n(R)$.

The case $k = 1$ follows from corollary 4.18.

Assume that $(\delta_k(R))^+ \subseteq \delta_k(\mathbb{M}_n(R))$ for some k.

Since $\delta_{k+1}(R) = \delta_1(\delta_k(R))$, therefore

$$
\begin{aligned}
(\delta_{k+1}(R))^+ &= (\delta_1(\delta_k(R)))^+ \\
&\subseteq \delta_1(\delta_k(R))^+), \text{ by corollary 4.19 and result 1.62,} \\
&\subseteq \delta_1(\delta_k(\mathbb{M}_n(R))), \text{ by the induction hypothesis,} \\
&= \delta_{k+1}(\mathbb{M}_n(R)).
\end{aligned}
$$

This completes the proof by induction.

<u>lemma 4.22</u>: Let I_1 and I_2 be ideals of a d.g. near-ring R. Then $(I_1^*, I_2^*) \subseteq (I_1, I_2)^*$.

Proof: Let $(\mathcal{A}, \mathcal{B}) \in (I_1^*, I_2^*)$, where $\mathcal{A} \in I_1^*$ and $\mathcal{B} \in I_2^*$. Take $\alpha \in R^n$.

$$
\begin{aligned}
\text{Then}\quad (\mathcal{A},\mathcal{B})\alpha &= (-\mathcal{A} - \mathcal{B} + \mathcal{A} + \mathcal{B})\alpha \\
&= -\mathcal{A}\alpha - \mathcal{B}\alpha + \mathcal{A}\alpha + \mathcal{B}\alpha \\
&= (\mathcal{A}\alpha, \mathcal{B}\alpha) \\
&\in (I_1^n, I_2^n), \text{ by definition of } I^*, \\
&= (I_1, I_2)^n, \text{ by result 1.5.}
\end{aligned}
$$

So $(\mathcal{A}, \mathcal{B}) \in (I_1, I_2)^*$.

Now let $\mathcal{U} \in (I_1^*, I_2^*)$. Then $\mathcal{U} = \varepsilon_1\mathcal{C}_1 + \varepsilon_2\mathcal{C}_2 + \ldots + \varepsilon_m\mathcal{C}_m$ where $\varepsilon_t = \pm 1$, $\mathcal{C}_t = (\mathcal{A}_t, \mathcal{B}_t)$, $\mathcal{A}_t \in I_1^*$, $\mathcal{B}_t \in I_2^*$, and $1 \leqslant t \leqslant m$.

Since $\mathcal{C}_t \in (I_1, I_2)^*$ for all t, $1 \leqslant t \leqslant m$, therefore $\mathcal{U} \in (I_1, I_2)^*$.

This completes the proof.

<u>Corollary 4.23</u>: Let R be a d.g. near-ring. Then

$\delta_1(\mathbb{M}_n(R)) \subseteq (\delta_1(R))^*$.

$$
\begin{aligned}
\text{Proof: } \delta_1(\mathbb{M}_n(R)) &= (\mathbb{M}_n(R), \mathbb{M}_n(R)) \\
&= (R^*, R^*) \\
&\subseteq (R,R)^*, \text{ by lemma 4.22,} \\
&= (\delta_1(R))^*.
\end{aligned}
$$

<u>Corollary 4.24</u>: Let R be a d.g. near-ring, and I be an ideal of R.

Then $\delta_1(I^*) \subseteq (\delta_1(I))^*$

Proof: This can be proved in a similar manner to the above result.

Corollary 4.25: Let R be a d.g. near-ring with identity. Then

$(\mathbb{M}_n(R); I_1^*, I_2^*) \subseteq (R; I_1, I_2)^*$.

Proof: The result follows from result 1.68, result 2.5 and lemma 4.22.

Now by corollaries 4.23 and 4.24 and using exactly the same technique as for theorem 4.21, we can prove the next result.

Theorem 4.26: Let R be a d.g. near-ring. Then

$(\delta_k(R))^* \supseteq \delta_k(\mathbb{M}_n(R))$ for all $k \geqslant 0$.

By result 1.69, we get the relationship between the distributor series of R and that of $\mathbb{M}_n(R)$.

Theorem 4.27: Let R be a d.g. near-ring with identity. Then

(1) $(D^k(R))^+ \subseteq D^k(\mathbb{M}_n(R))$ for all $k \geqslant 0$.

(2) $(D^k(R))^* \supseteq D^k(\mathbb{M}_n(R))$ for all $k \geqslant 0$.

We now wish to show the relationship between the lower central series of R and that of $\mathbb{M}_n(R)$.

Theorem 4.28: Let R be a d.g. near-ring with identity. Then

(1) $(\gamma_{k+1}(R))^+ \subseteq \gamma_{k+1}(\mathbb{M}_n(R))$ for all $k \geqslant 0$.

(2) $(\gamma_{k+1}(R))^* \supseteq \gamma_{k+1}(\mathbb{M}_n(R))$ for all $k \geqslant 0$.

Proof: (1) We use induction on k. The case $k = 0$ is obvious.

Let $k = 1$. Since $(\ )^+$ is an injection, so

$$(\gamma_2(R))^+ = (\delta_1(R))^+$$
$$\subseteq \delta_1(\mathbb{M}_n(R)), \text{ by corollary 4.18}$$

$= \gamma_2(\mathbb{M}_n(R))$

We next assume that $(\gamma_{k+1}(R))^+ \subseteq \gamma_{k+1}(\mathbb{M}_n(R))$. Then

$$
\begin{aligned}
(\gamma_{k+2}(R))^+ &= (R,\gamma_{k+1}(R))^+ \\
&\subseteq (R^+,(\gamma_{k+1}(R))^+), \text{ by lemma 4.17 and result 1.62,} \\
&= (\mathbb{M}_n(R),(\gamma_{k+1}(R))^+), \text{ as } R^+ = \mathbb{M}_n(R), \\
&\subseteq (\mathbb{M}_n(R),\gamma_{k+1}(\mathbb{M}_n(R))), \text{ by the induction hypothesis,} \\
&= \gamma_{k+2}(\mathbb{M}_n(R)).
\end{aligned}
$$

The result follows by induction.

The proof of part (2) is similar to that of part (1), and is therefore omitted.

It turns out that corollary 3.28 (1) and (2) can be extended in the d.g. case. To justify our claim, we need the following lemmas.

<u>Lemma 4.29</u>: Let R be a d.g. near-ring with identity, and let I be an ideal of R. If $I \subseteq Z(R)$, the centre of R, then $I^* \in Z(\mathbb{M}_n(R))$.

Proof: We need to show that $(\mathbb{M}_n(R),I^*) = \mathbf{0}$. Since $R^* = \mathbb{M}_n(R)$ and $(R^*,I^*) \subseteq (R,I)^*$, by lemma 4.22, therefore $(\mathbb{M}_n(R),I^*) \subseteq (R,I)^* = \{0\}^*$, as $(R,I) = \{0\}$, by the hypothesis, and $(\ )^*$ is an injection (see result 2.18). Now since $\{0\}^* = \mathbf{0}$, we get the result that we want.

<u>Lemma 4.30</u>: Let R be a d.g. near-ring with identity, and I be an ideal of R. If $I \subseteq Z(R)$, then $I^+ \subseteq Z(\mathbb{M}_n(R))$.

Proof: The result follows from the previous corollary as $I^+ \subseteq I^*$.

The converse of lemma 4.29 and that of lemma 4.30 can also be proved.

<u>Theorem 4.31</u>: Let R be a d.g. near-ring with identity, and I be an ideal of R. Then $I \subseteq Z(R)$ if and only if $I^+ \subseteq Z(\mathbb{M}_n(R))$.

Proof: Only the converse needs proof.

Let $I^+ \subseteq Z(\mathbb{M}_n(R))$. We have

$(I,R)^+ \subseteq (I^+,R^+)$, by lemma 4.17,

$= (I^+,\mathbb{M}_n(R))$, as $\mathbb{M}_n(R) = R^+$,

$= \mathbf{0}$, by hypothesis,

$= \{0\}^+$, as $\{0\}^+ = \mathbf{0}$.

Now since $(\ )^+$ is an injection (see result 2.18), therefore $(I,R) \subseteq \{0\}$. Hence $I \subseteq Z(R)$.

<u>Theorem 4.32</u>: Let R be a d.g. near-ring with identity, and I be an ideal of R. Then $I \subseteq Z(R)$ if and only if $I^* \subseteq Z\ (\mathbb{M}_n(R))$.

Proof: Only the converse needs proof.

Let $I^* \subseteq Z(\mathbb{M}_n(R))$. Since $I^+ \subseteq I^*$ (see result 2.10), therefore $I^+ \subseteq Z(\mathbb{M}_n(R))$. Hence $I \subseteq Z(R)$, by the above result.

We finish this chapter with the result which gives equality between ideals, I^+ and I^*, of $\mathbb{M}_n(R)$.

It is also remarked that there is still some work to be done in looking for weaker conditions that ensure that $I^+ = I^*$.

<u>Theorem 4.33</u>: Let R be a d.g. near-ring with identity. and I be an ideal in R. If $I^+ \supseteq \delta_1(\mathbb{M}_n(R))$, then $I^+ = I^*$.

Proof: Let $I^+ \supseteq \delta_1(\mathbb{M}_n(R)) \supseteq (\delta_1(R))^+$, by corollary 4.18.

Since $(\ )^+$ is an injection (see result 2.18), therefore $I \supseteq \delta_1(R)$. Now let $\mathcal{A} \in I^*$. Then

$$\begin{aligned}\mathcal{A} = {} & [a_1;1,1] + [b_1;1,2] + \ldots\ldots + [c_1;1,n] + \ldots\ldots + \\ & [a_m;1,1] + [b_m;1,2] + \ldots\ldots + [c_m;1,n] + \ldots\ldots + \\ & [x_1;n,1] + [y_1;n,2] + \ldots\ldots + [z_1;n,n] + \ldots\ldots + \\ & [x_q;n,1] + [y_q;n,2] + \ldots\ldots + [z_q;n,n]\end{aligned}$$

by an extension of result 2.33,

$$\begin{aligned} = {} & [a_1;1,1] + \ldots\ldots + [a_m;1,1] + [b_1;1,2] + \ldots\ldots + [b_m;1,2] + \\ & \ldots\ldots\ldots\ldots\ldots\ldots + [c_1;1,n] + \ldots\ldots + [c_m;1,n] + \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \\ & [x_1;n,1] + \ldots\ldots + [x_q;n,1] + [y_1;n,2] + \ldots\ldots + [y_q;n,2] + \\ & \ldots\ldots\ldots\ldots\ldots\ldots + [z_1;n,n] + \ldots\ldots + [z_q;n,n] + \\ & \mathcal{B}\end{aligned}$$

where $\mathcal{B} \in \delta_1(\mathbb{M}_n(R))$ (see definition 1.6),

$$\begin{aligned} = {} & [\sum_{t=1}^{m} a_t;1,1] + [\sum_{t=1}^{m} b_t;1,2] + \ldots\ldots\ldots\ldots + [\sum_{t=1}^{m} c_t;1,n] + \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \\ & [\sum_{l=1}^{q} x_l;n,1] + [\sum_{l=1}^{q} y_l;n,2] + \ldots\ldots\ldots\ldots + [\sum_{l=1}^{q} z_l;n,n] + \\ & \mathcal{B}.\end{aligned}$$

Now since $\sum_{t=1}^{m} a_t$, $\sum_{t=1}^{m} b_t$, $\sum_{t=1}^{m} c_t, \ldots\ldots,$ $\sum_{l=1}^{q} x_l$, $\sum_{l=1}^{q} y_l$, $\sum_{l=1}^{q} z_l \in I$, by an extension of result 2.33, and $\mathscr{B} \in I^+$, therefore $\mathscr{A} \in I^+$, by definition of I^+.

Combining this fact with result 2.10, we get $I^+ = I^*$.

CHAPTER 5

GENERALIZED-DISTRIBUTIVE MATRIX NEAR-RINGS

In this chapter, we shall use some of our previously developed theory to establish certain facts concerning generalized-distributivity in the base near-ring R and the matrix near-ring $\mathbb{M}_n(R)$.

It turns out that weakly-distributive matrix near-rings with identity are closely related to matrix rings. For example, we show that, like the ring case, if L is a maximal left ideal of a w.d. near-ring R and $\alpha \in R^n - L^n$, then $(L^n:\alpha)$ is maximal in $\mathbb{M}_n(R)$ (see result 2.38 and theorem 5.15).

It is to be noted that we discuss only weakly-distributive d.g. near-rings here.

We start with a result which follows from result 1.70 and corollary 3.9.

Theorem 5.1: Let R be a d.g. near-ring with identity. Then R is weakly-distributive if and only if $\mathbb{M}_n(R)$ is weakly-distributive.

Proof: R is weakly-distributive

if and only if (R,+) is soluble, by result 1.70,

if and only if $(\mathbb{M}_n(R),+)$ is soluble, by corollary 3.9, if and only if $\mathbb{M}_n(R)$ is weakly-distributive, by result 1.70, as $\mathbb{M}_n(R)$ is d.g. and has an identity element.

Note that this result can also be seen in Abbasi, Meldrum and Meyer [1].

Theorem 5.2: Let R be a d.g. near-ring with identity. Then R is $\bar{d}$.w.d. if and only if $\mathbb{M}_n(R)$ is $\bar{d}$.w.d.

Proof: R is $\bar{d}$.w.d.

if and only if (R,+) is $\bar{c}$-nilpotent, by result 1.93,

if and only if $(\mathbb{M}_n(R),+)$ is $\bar{c}$-nilpotent, by corollary 3.13,

if and only if $\mathbb{M}_n(R)$ is $\bar{d}$.w.d., by result 1.93, as $\mathbb{M}_n(R)$ is d.g. and has an identity element.

Theorem 5.3: Let R be a d.g. near-ring with identity. Then R is $\underline{d}$.w.d. if and only if $\mathbb{M}_n(R)$ is $\underline{d}$.w.d.

Proof: R is $\underline{d}$.w.d.

if and only if R is $\bar{d}$.w.d., by result 1.82,

if and only if $\mathbb{M}_n(R)$ is $\bar{d}$.w.d., by theorem 5.2,

if and only if $\mathbb{M}_n(R)$ is $\underline{d}$.w.d., by result 1.82, as $\mathbb{M}_n(R)$ is d.g.

It is obvious that $R^2 = R$, if (R,+) is a connected R-module. This fact together with result 1.70 gives us the following lemma.

Lemma 5.4: Let (R,+) be a connected R-module. Then (R,+) is soluble if and only if R is weakly-distributive.

This result enables us to extend theorem 5.1.

Theorem 5.5: If $(R,+)$ is a connected R-module, then R is weakly-distributive if and only if $\mathbb{M}_n(R)$ is weakly-distributive.

Proof: Let R be a weakly-distributive near-ring. Since $(R,+)$ is soluble (see lemma 5.4), so $(R^n,+)$ is soluble, by result 1.12. Now as $(R^n,+)$ is a connected $\mathbb{M}_n(R)$- module, by result 2.11, therefore $\mathbb{M}_n(R)$ is weakly-distributive, by lemma 5.4.

Conversely, let $\mathbb{M}_n(R)$ be a weakly-distributive near-ring. Since $(R^n,+)$ is a connected $\mathbb{M}_n(R)$-module, by result 2.11, therefore $(R^n,+)$ is soluble, by lemma 5.4. This forces $(R,+)$ to be soluble. Hence R is weakly-distributive, by lemma 5.4.

Remark: If $(R,+)$ is a monogenic R-module, then we have the analogous results to lemma 5.4 and theorem 5.5, as every monogenic R-module is connected.

Corollary 5.6: If R is ν-primitive on $(R,+)$, for $\nu = 0, 1, 2$, then R is weakly-distributive if and only if $\mathbb{M}_n(R)$ is weakly-distributive.

Proof: The result follows immediately from theorem 5.5 and the remark made earlier as $(R,+)$ is a monogenic R-module (see definition 1.36).

Lemma 5.7: Let R be a prime d.g. near-ring. If $(R,+)$ is soluble, then R is a ring.

Proof: It is known that, if R is a d.g. near-ring such that $(R,+)$ is soluble then $\delta_1(R)$ is multiplicatively nilpotent (see result 1.71). This forces $\delta_1(R) = \{0\}$, as a prime near-ring can not have any nilpotent subset other than $\{0\}$ (see definition 1.51). This means that $(R,+)$ is abelian and hence is a ring, by result 1.58.

<u>Theorem</u> <u>5.8</u>: Let R be a prime weakly-distributive near-ring with $R^2 = R$. Then R is a ring.

Proof: The result follows from lemma 5.7 and result 1.70.

<u>Theorem</u> <u>5.9</u>: Let R be a ν-primitive weakly-distributive near-ring with $R^2 = R$ for $\nu = 0, 1, 2$. Then R is a ring.

Proof: As R is prime, by result 1.52, so the result follows immediately from theorem 5.8.

<u>Theorem</u> <u>5.10</u>: Let R be a ν-primitive weakly-distributive near-ring on $(R,+)$, for $\nu = 0, 1, 2$. Then R is a ring.

Proof: Since $(R,+)$ is a monogenic R-module, therefore $R^2 = R$. Now we have our result by theorem 5.9.

<u>Theorem</u> <u>5.11</u>: Let R be a ν-primitive weakly-distributive near-ring on $(R,+)$ for $\nu = 0, 1, 2$. Then $\mathbb{M}_n(R)$ is a ring.

Proof: The result follows from theorem 5.10 and corollary 4.11.

Our next task is to solve two open problems posed by J. H. Meyer [8]:

(1) Find a relationship between $(J_{1/2}(R))^*$ and $J_{1/2}(\mathbb{M}_n(R))$ (see problem 8).

(2) If L is a maximal ideal in R and $\alpha \in R^n - L^n$, then show that $(L^n:\alpha)$ is maximal in $\mathbb{M}_n(R)$ (see problem 4).

We first describe the relationship between $(J_\nu(R))^*$ and $J_\nu(\mathbb{M}_n(R))$ for $\nu = 0, 1/2$.

The following result is an immediate consequence of result 1.70 and result 1.75.

Lemma 5.12: Let R be a weakly-distributive near-ring with identity. Then all J_ν-radicals coincide for $\nu = 0, 2$.

Theorem 5.13: If R is a weakly-distributive near-ring with identity. Then $(J_0(R))^* = J_0(\mathbb{M}_n(R))$.

Proof: Since $J_0(R) = J_2(R)$, by lemma 5.12, and $(\ )^*$ is an injection (see result 2.18), therefore

$$\begin{aligned}(J_0(R))^* &= (J_2(R))^* \\ &= J_2(\mathbb{M}_n(R)), \text{ by result 2.32,} \\ &= J_0(\mathbb{M}_n(R)),\end{aligned}$$

by lemma 5.12, as $\mathbb{M}_n(R)$ is a weakly-distributive near-ring with identity (see theorem 5.1).

Theorem 5.14: If R is a weakly-distributive near-ring with identity, then $(J_{1/2}(R))^* = J_{1/2}(\mathbb{M}_n(R))$.

Proof: Since $J_0(R) \subseteq J_{1/2}(R) \subseteq J_2(R)$ (see result 1.49) and $J_0(R) = J_2(R)$, by lemma 5.10, therefore $J_0(R) = J_{1/2}(R)$. Now the rest of the proof goes exactly like that of theorem 5.13 and is therefore omitted.

Note that this work can also be seen in Abbasi, Meldrum and Meyer [1].

Next we solve another open problem.

Theorem 5.15: Let R be a weakly-distributive near-ring with identity. If L is a maximal left ideal in R and $\alpha \in R^n - L^n$, then $(L^n:\alpha)$ is maximal in $\mathbb{M}_n(R)$.

Proof: Since all maximal left ideals of R are strictly maximal ideals, by result 1.70 and result 1.74, therefore L is a strictly maximal left ideal of R. So $(L^n:\alpha)$ is a strictly maximal left ideal of $\mathbb{M}_n(R)$ (see result 2.38) and hence a maximal left ideal of $\mathbb{M}_n(R)$.

The foregoing results show the close relationship between matrix near-rings and weakly-distributive matrix near-rings. We continue to generalize results from matrix rings to these matrix near-rings.

It is well known that, for a maximal ideal M of a ring R and $\alpha \in R^n - M^n$, we have a maximal ideal $(M^n:\alpha)$ of the matrix ring $M_n(R)$ and if X is any matrix in $M_n(R) - (M^n:\alpha)$, then $((M_n:\alpha):X)$ is a maximal left ideal of $M_n(R)$. In fact, $((M^n:\alpha):X) = (M^n:X\alpha)$ (see example 1.4, Stone [12]).

We show that this result is also true if R is a weakly-distributive near-ring with identity.

Theorem 5.16: Let $\mathfrak{X} \in \mathbb{M}_n(R) - (L^n:\alpha)$. This implies that $\mathfrak{X}\alpha \notin L^n$. So $(L^n:\mathfrak{X}\alpha)$ is a maximal ideal of $\mathbb{M}_n(R)$, by theorem 5.15. We aim to show that $((L^n:\alpha):\mathfrak{X}) = (L^n:\mathfrak{X}\alpha)$.

Let $\mathcal{A} \in ((L^n:\alpha):\mathcal{X})$. This implies that $\mathcal{A}\mathcal{X} \in (L^n:\alpha)$. This implies that $\mathcal{A}\mathcal{X}\alpha \in L^n$. This implies that $\mathcal{A} \in (L^n:\mathcal{X}\alpha)$.

Now let $\mathcal{B} \in (L^n:\mathcal{X}\alpha)$. This implies that $\mathcal{B}\mathcal{X}\alpha \in L^n$. This implies that $\mathcal{B}\mathcal{X} \in (L^n:\alpha)$. This implies that $\mathcal{B} \in ((L^n:\alpha):\mathcal{X})$. Therefore $(L^n:\mathcal{X}\alpha) = ((L^n:\alpha):\mathcal{X})$ and hence $((L^n:\alpha):\mathcal{X})$ is a maximal ideal of $\mathbb{M}_n(R)$.

Stone [12] showed that if L is a maximal left ideal of a ring R and $\alpha, \beta \in R^n - L^n$, then $(L^n:\alpha) = (L^n:\beta)$ if $\alpha \equiv \beta \pmod{L^n}$. Moreover, $(L^n:\alpha) = (L^n:\beta)$ if $\beta \equiv \alpha c \pmod{L^n}$, where $c \in I(L)$, the idealizer of L.

We need the following result to show the analogue of these results for weakly-distributive matrix near-rings.

Lemma 5.17: Let R be a weakly-distributive near-ring with identity. If L is a maximal left ideal of R, then $L \supseteq D^1(R)$.

Proof: Recall that $J_{1/2}(R)$ contains all nil ideals (see results 1.49 and 1.50), $D^1(R)$ is nilpotent (see result 1.72) and if R is a near-ring with identity, then $J_{1/2}(R) = \cap \{L : L \text{ is a maximal left ideal of } R\}$ (see definitions 1.46 and 1.48). Therefore $L \supseteq J_{1/2}(R) \supseteq D^1(R)$, as a nilpotent subset is necessarily nil.

Theorem 5.18: Let R be a weakly-distributive near-ring with identity, L be a maximal left ideal of R, and $\alpha, \beta \in R^n - L^n$. Then

$(L^n:\alpha) = (L^n:\beta)$ if $\alpha \equiv \beta \pmod{L^n}$.

Proof: Let $\alpha \equiv \beta \pmod{L^n}$, that is, $\beta - \alpha \in L^n$. So $\beta = \alpha + \mu$, where $\mu \in L^n$. Let $\mathcal{A} \in (L^n:\alpha)$. We need to show that $\mathcal{A} \in (L^n:\beta)$. We have

$\mathcal{A}\beta = \mathcal{A}(\alpha + \mu) = \mathcal{A}\alpha + \mathcal{A}\mu + \sigma$, where $\sigma \in (D^1(R))^n$, by lemma 4.13. Since $\mathcal{A}\alpha \in L^n$ (see definition 1.25), $\mathcal{A}\mu \in L^n$, as L^n is a left $\mathbb{M}_n(R)$-ideal of R^n (see result 2.6) and $\sigma \in L^n$, as $D^1(R) \subseteq L$ (see lemma 5.17), therefore $\mathcal{A}\beta \in L^n$. So $\mathcal{A} \in (L^n:\beta)$. Hence $(L^n:\alpha) \subseteq (L^n:\beta)$. Similarly, we can show that $(L^n:\beta) \subseteq (L^n:\alpha)$, as $\alpha \equiv \beta \pmod{L^n}$ if and only if $\beta \equiv \alpha \pmod{L^n}$. This completes the proof.

For our next result, we first make the folowing definition.

Definition 5.19: Let R be a right near-ring, and L be a left ideal of R. $I(L)$, the idealizer of L, is defined as

$$I(L) := \{r \in R : Lr \subseteq L\}.$$

Obviously, if L is a two-sided ideal of R, then $I(L) = R$.

Theorem 5.20: Let R be a weakly-distributive near-ring with identity, L be a maximal left ideal of R and $\alpha, \beta \in R^n - L^n$. Then $(L^n:\alpha) = (L^n:\beta)$ if $\beta \equiv \alpha c \pmod{L^n}$, where $c \in I(L)$.

Proof: It is sufficient to show that $(L^n:\alpha) \subseteq (L^n:\beta)$, as $(L^n:\alpha)$ and $(L^n:\beta)$ are maximal left ideals of $\mathbb{M}_n(R)$ (see theorem 5.15).

If $\beta \equiv \alpha c \pmod{L^n}$, then $\beta = \alpha c + \mu$, where $\mu \in L^n$.

Let $\mathscr{A} \in (L^n:\alpha)$. We must show that $\mathscr{A} \in (L^n:\beta)$. We have $\mathscr{A}\beta = \mathscr{A}(\alpha c + \mu) = \mathscr{A}(\alpha c) + \mathscr{A}\mu + \sigma$, where $\sigma \in (D^1(R))^n$, by lemma 4.13. Since c is an $\mathbb{M}_n(R)$-endomorphism of R^n (see result 2.39), so $\mathscr{A}\beta = (\mathscr{A}\alpha)c + \mathscr{A}\mu + \sigma$. Now as $\mathscr{A}\alpha \in L^n$, therefore $\pi_i(\mathscr{A}\alpha) \in L$, for all i, $1 \leqslant i \leqslant n$, and $c \in I(L)$ forces $\pi_i(\mathscr{A}\alpha)c \in L$. That is, $(\mathscr{A}\alpha)c \in L^n$. We have already shown in the proof of theorem 5.18 that $\mathscr{A}\mu$ and σ are in L^n. Hence we get $\mathscr{A}\beta \in L^n$, which means that $\mathscr{A} \in (L^n:\beta)$.

SOME OPEN PROBLEMS

Let R be a d.g. near-ring.

(1) Is $(\delta_1(R))^*$ equal to $\delta_1(\mathbb{M}_n(R))$ (see corollary 4.23)?

(2) Is $(\delta_1(R))^+$ equal to $\delta_1(\mathbb{M}_n(R))$ (see corollary 4.18)?

(3) Is $(I_1, I_2)^+$ equal to (I_1^+, I_2^+) where I_1 and I_2 are ideals in R see lemma 4.17)?

(4) Is $(I_1, I_2)^*$ equal to (I_1^*, I_2^*) where I_1 and I_2 are ideals in R (see lemma 4.22)?

(5) Find weaker conditions that ensure that $I^+ = I^*$ where I is an ideal in R (see theorem 4.33).

(6) Is $\overline{\mathbb{M}}_n(I)$ an ideal in $\mathbb{M}_n(R)$, as in the ring case?

(7) Is $J(\mathbb{M}_n(R))$ equal to $\overline{\mathbb{M}}_n(J(R))$, as in the ring case (see lemmas 3.40, 5.12 and theorem 5.13)?

(8) Can theorems 4.31 and 4.32 be extended for non d.g. near-rings?

REFERENCES

[1] Abbasi, S. J., Meldrum, J. D. P. and Meyer, J. H., A note on the J_0-radical in matrix near-rings, submitted.

[2] Fröhlich, A., Distributively-generated near-rings. 1. Ideal Theory. Proc. London Math. Soc.(3)8, 76-94, 1958.

[3] McCoy, N. H., Rings and Ideals, The Carus Mathematical Monographs, (8). The Mathematical Association of America, 1948.

[4] McCoy, N. H., The Theory of Rings. The Macmillan Company, New York, 1964.

[5] Meldrum, J. D. P., Near-rings and their links with groups, Research notes in Mathematics, Pitman, 1985.

[6] Meldrum, J. D. P. and Van Der Walt, A. P. J., Matrix Near-rings, Arch. Math. (Basel), Vol. 47, 312-319, 1986.

[7] Meyer, J. H. and Van Der Walt, A. P. J., Solution of an open problem concerning 2-primitive near-rings, Near-rings and near-fields (ed. G. Betsch), North Holland, Amsterdam, 185-192, 1987.

[8] Meyer J. H., Matrix near-rings, Ph.D. Dissertation, University of Stellenbosch, South Africa, 1986.

[9] Pilz, G., Near-rings, Revised Edition, North Holland, 1983.

[10] Roberts, I. G. T., Generalized-distributivity in near-rings, M.Phil. Dissertation, University of Edinburgh, U.K., 1983.

[11] Robinson, D. J. S., A course in the theory of groups, Graduate Texts in Mathematics, Spinger-Verlag, New York, 1980.

[12] Stone, D. R., Maximal left ideals and idealizers in matrix rings. Can. J. Math., Vol. 32, No. 6, 1397-1410, 1980.

[13] Van Der Walt, A. P. J., Primitivity in matrix near-rings. Quaest. Math., 9, 459-469, 1986.

[14] Van Der Walt, A. P. J., On two-sided ideals in matrix near-rings, Near-rings and near-fields (ed.: G. Betsch), North-Holland, Amsterdam, 267-272, 1987.